Analog to Digital Conversion
Oversampling or Σ-Δ converters

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Outline of the Lecture

Analog to Digital Conversion
- Oversampling or $\Sigma$-$\Delta$ converters

Sampling Theorem - Quantization of Signals

PCM (Successive Approximation, Flash ADs)

DPCM - DELTA - $\Sigma$-$\Delta$ Modulation
Principles - Characteristics
1st and 2nd order
Noise shaping - performance

Decimation and A/D converters

Filtering of $\Sigma$-$\Delta$ sequences
Analog & digital signals

**Analog**

*Continuous function* $V$ of *continuous* variable $t$ (time, space etc): $V(t)$.

**Digital**

*Discrete function* $V_k$ of *discrete* sampling variable $t_k$, with $k = \text{integer}$: $V_k = V(t_k)$.

Uniform (periodic) sampling.
Sampling frequency $f_S = 1/ \tau_S$
AD Conversion - Details
Sampling

(a) Analog frequency = 0.0 (i.e., DC)

(b) Analog frequency = 0.09 of sampling rate

(c) Analog frequency = 0.31 of sampling rate

(d) Analog frequency = 0.95 of sampling rate
How fast do we have to instantly stare at the disk if it rotates with frequency 0.5 Hz?
The sampling theorem

A signal $s(t)$ with maximum frequency $f_{\text{MAX}}$ can be recovered if sampled at frequency $f_s > 2 f_{\text{MAX}}$.

* Multiple proposers: Whittaker(s), Nyquist, Shannon, Kotelnikov.

Naming gets confusing!

Nyquist frequency (rate) $f_N = 2 f_{\text{MAX}}$ or $f_{\text{MAX}}$ or $f_{\text{S,MIN}}$ or $f_{\text{S,MIN}}/2$

Example

$s(t) = 3 \cdot \cos(50\pi t) + 10 \cdot \sin(300\pi t) - \cos(100\pi t)$

Condition on $f_s$?

$F_1 = 25$ Hz, $F_2 = 150$ Hz, $F_3 = 50$ Hz

$f_{\text{MAX}}$

$f_s > 300$ Hz
Sampling and Spectrum

(a) $x(t)$

(b) $X(f)$

(c) $P(t)$

(d) $X'(f)$

$0 \quad 1 \quad 2F_s \quad -2F_s \quad -F_s \quad F_s \quad 2F_s \quad f$

$t$

$t$
Sampling low-pass signals

(a) Band-limited signal: frequencies in [-B, B] \((f_{\text{MAX}} = B)\).

(b) Time sampling \(\rightarrow\) frequency repetition.
\[f_s > 2B \quad \rightarrow \quad \text{no aliasing.}\]

(c) \(f_s \leq 2B \quad \rightarrow \quad \text{aliasing!}\)

\textbf{Aliasing: signal ambiguity in frequency domain}
Quantization and Coding

N Quantization Levels

Quantization Noise
**SNR of ideal ADC**

\[
\begin{align*}
\text{SNR}_{\text{ideal}} &= 20 \cdot \log_{10} \left( \frac{\text{RMS}(\text{input})}{\text{RMS}(e_q)} \right) \\
\end{align*}
\]

(1)

Also called SQNR

(signal-to-quantisation-noise ratio)

**Assumptions**

- Ideal ADC: only quantisation error \( e_q \)
  
  (p(e) constant, no stuck bits…)

- \( e_q \) uncorrelated with signal
  
  (noise spectrum???)

- ADC performance constant in time.

\[
\text{RMS}(\text{input}) = \sqrt{\frac{1}{T} \int_0^T \left( \frac{V_{\text{FSR}}}{2} \cdot \sin(\omega t) \right)^2 dt} = \frac{V_{\text{FSR}}}{2\sqrt{2}}
\]

\[
\text{RMS}(e_q) = \sqrt{\frac{q/2}{-q/2} \int e_q^2 \cdot p(e_q) de_q} = \frac{q}{\sqrt{12}} = \frac{V_{\text{FSR}}}{2^N \cdot \sqrt{12}}
\]

(sampling frequency \( f_S = 2 f_{\text{MAX}} \))

**Input(t)**

\[ \text{Input}(t) = \frac{1}{2} V_{\text{FSR}} \sin(\omega t). \]

\[ p(e) \]

quantisation error probability density

\[ \frac{1}{q} \quad -\frac{q}{2} \quad \frac{q}{2} \quad e_q \]

Error value
Substituting in (1):

$$SNR_{\text{ideal}} = 6.02 \cdot N + 1.76 \text{ [dB]}$$ \hspace{1cm} (2)

One additional bit $\Rightarrow$ SNR increased by 6 dB

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Real SNR lower because:

- Real signals have noise.
- Forcing input to full scale unwise.
- Real ADCs have additional noise (aperture jitter, non-linearities etc).

Actually (2) needs correction factor depending on ratio between sampling freq & Nyquist freq. Processing gain due to oversampling.
Coding – Flash AD
Coding - Conventional
Digital Filters

- They are characterized by their Impulse Response $h(n)$, their Transfer Function $H(z)$ and their Frequency Response $H(\omega)$.
- They can have memory, high accuracy and no drift with time and temperature.
- They can possess linear phase.
- They can be implemented by digital computers.
IIR

\[
y(n) = \sum_{k=0}^{N} a(k)x(n-k) - \sum_{k=1}^{M} b_k y(n-k)
\]

\[
H(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 + \sum_{k=1}^{M} b_k z^{-k}}
\]
Digital Filters - Categories

\[ y(n) = \sum_{k=0}^{N} h(k)x(n - k) = \sum_{k=0}^{N} a(k)x(n - k) \]

\[ H(z) = \sum_{k=0}^{N} a_k z^{-k} = \sum_{k=0}^{N} h(k)z^{-k} \]

- Stable
- Linear phase
Digital Filters - Examples

\[ H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}} \]

- \( a_0 = 0.498, \quad a_1 = 0.927, \quad a_2 = 0.498, \quad b_1 = -0.674, \quad b_2 = -0.363 \)

\[ H(z) = \sum_{k=0}^{11} h(k) z^{-k} \]

- \( h(1) = h(10) = -0.04506 \)
- \( h(2) = h(9) = 0.06916 \)
- \( h(3) = h(8) = -0.0553 \)
- \( h(4) = h(7) = -0.06342 \)
- \( h(5) = h(4) = 0.5789 \)
FIR filters

\[ y(n) = \sum_{k=0}^{N} h(k)x(n - k) = \sum_{k=0}^{N} a(k)x(n - k) \]

\[ H(z) = \sum_{k=0}^{N} a_k z^{-k} = \sum_{k=0}^{N} h(k)z^{-k} \]

- Stable
- Linear phase

Design Methods
- Optimal filters
- Windows method
- Sampling frequency
If you shift in time one signal, then you have to shift the other signals the same amount of time in order the final wave remains unchanged. For faster signals the same time interval means larger phase difference.

Proportional to the frequency of the signals. -\( \alpha \omega \)
Oversampling and Noise Shaping Converters

- Since the increase in sampling rate results only in spreading the power of the quantization noise over a wider bandwidth, we gain only half a bit every doubling of the frequency.

\[
\text{SNR}_1 = \frac{N_s}{N_q} \quad \text{SNR}_2 = \frac{N_s}{N_q/2}
\]

or \( \text{SNR}_2 = \text{SNR}_1 + 3 \text{ dBs} \)

\[
\downarrow 1/2 \text{ bit}
\]

- Better efficiency is obtained when the in-band noise is reduced regardless of the out-of-band noise.
Quantization Noise Shaping

**Figure 6-9** Multi-Order Sigma-Delta Noise Shapers

**Figure 6-10** Spectra of Three Sigma-Delta Noise Shapers
Differential Pulse Code Modulation - DPCM

- It is actually a PCM technique in which the difference between the consecutive samples is coded.
- It has larger dynamic range but it is more sensitive to noise.

Delta Modulation - DM

- It is a one-bit DPCM

- The appearance of the +1 and -1 are related to the slope of the signal.
- High sampling frequency is required.
- The corresponding circuits are relative simple.
- It is preferred in noisy environment.
Delta Modulation – One bit DPCM

The simplest form is the exponential Delta Modulation (An RC circuit as integrator in the feedback loop).

The coder is an integrator as the one in the feedback loop.
Delta Sequences

- The simplest delta encoder

- The Appearance of positive and negative pulses is related to the slope of the signal.

- Constant value signal results to the Idling pattern $I_a$ (alternative positive and negative pulses).

![Graph showing delta sequences and their derivatives](image)
This encoder is used in order to avoid overload.
ΔΣ – Encoder Behavior - Signal Transfer Function

\[ Y(z) = X(z) \cdot z^{-1} + Y(z) \cdot z^{-1} \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1}} \]

Signal transfer function \((E=0)\)

\[ Y(z) = X(z) \cdot \frac{z^{-1}}{1 - z^{-1}} - Y(z) \cdot \frac{z^{-1}}{1 - z^{-1}} \Rightarrow \]

\[ Y(z) \left[1 + \frac{z^{-1}}{1 - z^{-1}}\right] = X(z) \cdot \frac{z^{-1}}{1 - z^{-1}} \Rightarrow \]

\[ S(z) = \frac{Y(z)}{X(z)} = z^{-1} = e^{j\omega T} \quad |S(z)| = 1 \]
ΔΣ – Encoder Behavior

Quantization Noise Signal Transfer Function (X=0)

\[
Y(z) = E(z) - Y(z) \frac{Z^{-1}}{1-Z^{-1}}
\]

\[
N(z) = \frac{Y(z)}{E(z)} = (1-Z^{-1}) \quad \text{High-pass filter}
\]

i.e.

\[
|N(z)| = \sqrt{(1-\cos \omega T_s)^2 + \sin^2 \omega T_s}
\]

\[
= \sqrt{2 - 2 \cos \omega T_s}
\]

\[
= \sqrt{2 - 2(1-2\sin^2 \frac{\omega T_s}{2})} = \left| 2 \sin \frac{\omega T_s}{2} \right|
\]

Noise shaping transfer function of a first order modulator
Quantization Noise Power Density in the Signal Band

- **Original noise power density:**
  \[ S_q = \frac{\Delta^2}{12} \cdot \frac{1}{f_s} \]
  \( \rightarrow \) flat over the range \([-f_s/2, f_s/2]\)

- **Modified noise power density:**
  \[ S_N = \frac{\Delta^2}{12} \cdot \frac{1}{f_s} \left(2 \sin \frac{w T_s}{2}\right)^2 \]
  \( \rightarrow \) for the region \([-f_B, f_B]\) where
  \( f_B \ll f_s \)
  \[ \sin \frac{w T_s}{2} \approx \frac{w T_s}{2} \]

- **Total Noise power in the signal band:**
  \[ N_B = \int_{-f_B}^{f_B} \frac{\Delta^2}{12} \cdot \frac{1}{f_s} \left(w T_s\right)^2 \, df \]
  \[ = \frac{\Delta^2}{12} \cdot \frac{4 \pi^2}{f_s^3} \int_{-f_B}^{f_B} f^2 \, df \]
  \[ = K \cdot \frac{1}{f_s^3} \]

- Doubling the sampling rate we gain 9 dBs
  or \(1\frac{1}{2}\) bit resolution.

- We have to take care for the out-of-band noise.
A Second Order $\Delta\Sigma$ Modulator

- In a similar way we find

$$Y(z) = z^{-1}X(z) + (1-z^{-1})^2 E$$

- Total noise power

$$N_B = k \cdot \frac{1}{f_s^5}$$

$\rightarrow$ Doubling the frequency we gain 15 dBs or $2\frac{1}{2}$ bits resolution
Third Order Noise Shaper

Figure 8-1  Spectrum of a Third-Order Noise Shaper (16384 FFT bins)

flat response due to the FIR filter arithmetic rounding
Digital Decimation

- The out-of-band noise has been amplified.

So

→ We have to reduce the sampling rate at a value two times the signal frequency.

but

→ we have first to eliminate the out-of-band noise to avoid aliasing using low-pass filter(s) with many zeros (FIR filters)

Architecture of a typical digital decimator.

- Comb filters are preferable

\[
D(z) = \frac{1}{N} \left[ \frac{1-z^{-N}}{1-z^{-1}} \right]^K
\]
Digital Decimation – $\Sigma\Delta$ to PCM Conversion

**Figure 7-1** Digital Decimation Process

**Figure 7-3** Transfer Function of a Comb-Filter

$$y(n) = \sum_{i=0}^{15} x(n-i) : \text{Moving Average Process}$$

$$Y(z) = \frac{1 - z^{-16}}{1 - z^{-1}} X(z)$$
Digital Decimation – ΣΔ to PCM Conversion

Figure 8-1 Spectrum of a Third-Order Noise Shaper (16384 FFT bins)

Figure 7-7 FIR Filter Magnitude Response

Figure 7-8 Aliased Noise Bands of FIR Filter Output
Multi-Rate FIR filters

\[
\Delta f_1 = 0.1150 \\
\delta_{p1} = 0.0033 \\
\delta_{s1} = 0.001 \\
N_1 = 25
\]

\[
\Delta f_2 = 0.0875 \\
\delta_{p2} = 0.0033 \\
\delta_{s2} = 0.001 \\
N_2 = 34
\]

\[
\Delta f_3 = 0.025 \\
\delta_{p3} = 0.0033 \\
\delta_{s3} = 0.001 \\
N_3 = 117
\]
Low rate sequences are finally stored on CDs
Interpolation is used to convert the Low rate sequences stored on the CDs to analog signals. Oversampling is used.

Figure 8.3 Time domain illustration of interpolation by a factor of \( L = 3 \). Note that for each sample of \( x(n) \), three output samples \( y(m) \) are obtained.

Figure 8.4 Spectral interpretation of interpolation of a signal from 2 kHz to 6 kHz.
Oversampling in Audio Systems

Figure 8.18  Simplified block diagram of single-bit ADC scheme.

Figure 8.19  Audio signal reproduction in the compact disc system.