Basics on Digital Signal Processing

Transforms - DFT - FFT

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Outline of the Lecture

1. Orthogonal Transforms
2. Inverse Transforms
3. The Discreet Fourier Transform (DFT)
    - Graphical Derivation
    - Mathematical Derivation
4. Properties and complexity of the DFT
5. Examples
6. The Fast Fourier Transform

Orthogonal Transforms

$$x(t) = \sum_n a_n f_n(t)$$

Inverse

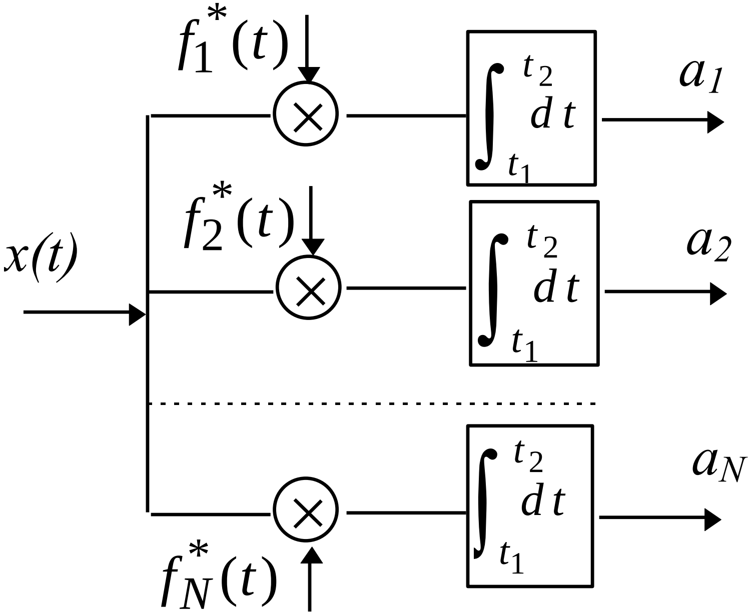
$$\int_{t_1}^{t_2} f_i(t) f_j^*(t)\, dt = \int_{t_1}^{t_2} f_i^*(t) f_j(t)\, dt = \begin{cases} 0 & i \neq j \\ c_i & i = j \end{cases}$$

Direct

$$a_i = \int_{t_1}^{t_2} x(t) f_i^*(t)\, dt =$$

$$= \int_{t_1}^{t_2} \Big(\sum_n a_n f_n(t)\Big) f_i^*(t)\, dt =$$

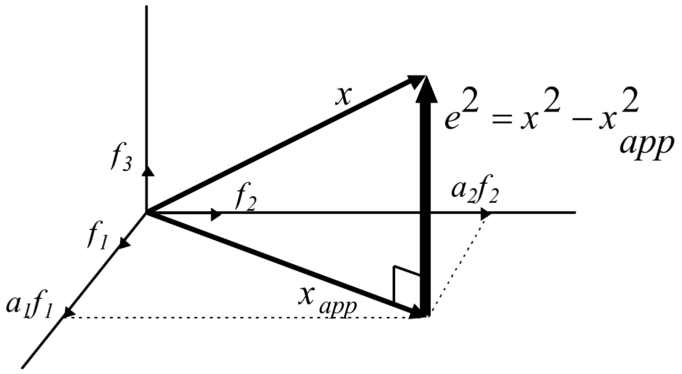
$$= \sum_n a_n \int_{t_1}^{t_2} f_n(t) f_i^*(t)\, dt$$



Orthogonal Transforms

$$x(t) = \sum_{i=0}^{\infty} a_i f_i(t)$$

$$x(t) \cong \sum_{i=0}^{N} a_i f_i(t) = x_{app}(t)$$



Orthogonality principle

For incomplete base

$f_i = e^{j\omega_i t}$

$f_k = e^{j\omega_k t}$ ⇒ **Fourier**

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Continuous Fourier Transform

$$x(t)=\begin{cases}1 & -t_1<t<t_1\\ 0 & \text{elsewhere}\end{cases}$$

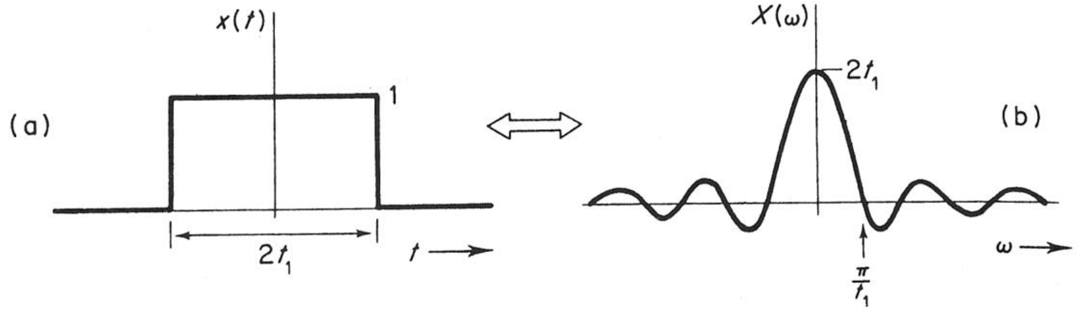
$$X(\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \iff x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt=$$

$$=\int_{-t_1}^{t_1} 1\cdot e^{-j\omega t}dt=-\frac{1}{j\omega}[e^{-j\omega t}]_{-t_1}^{t_1}=$$

$$=\frac{1}{j\omega}[e^{j\omega t}-e^{-j\omega t}]=$$

$$=2t_1\frac{\sin(\omega t_1)}{\omega t_1}$$



DFT - Mathematical Derivation

$$X(\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \Leftrightarrow \quad x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Continuous

$$x(t) \rightarrow x_D(n) \qquad X(\omega) \rightarrow X_D(k)$$

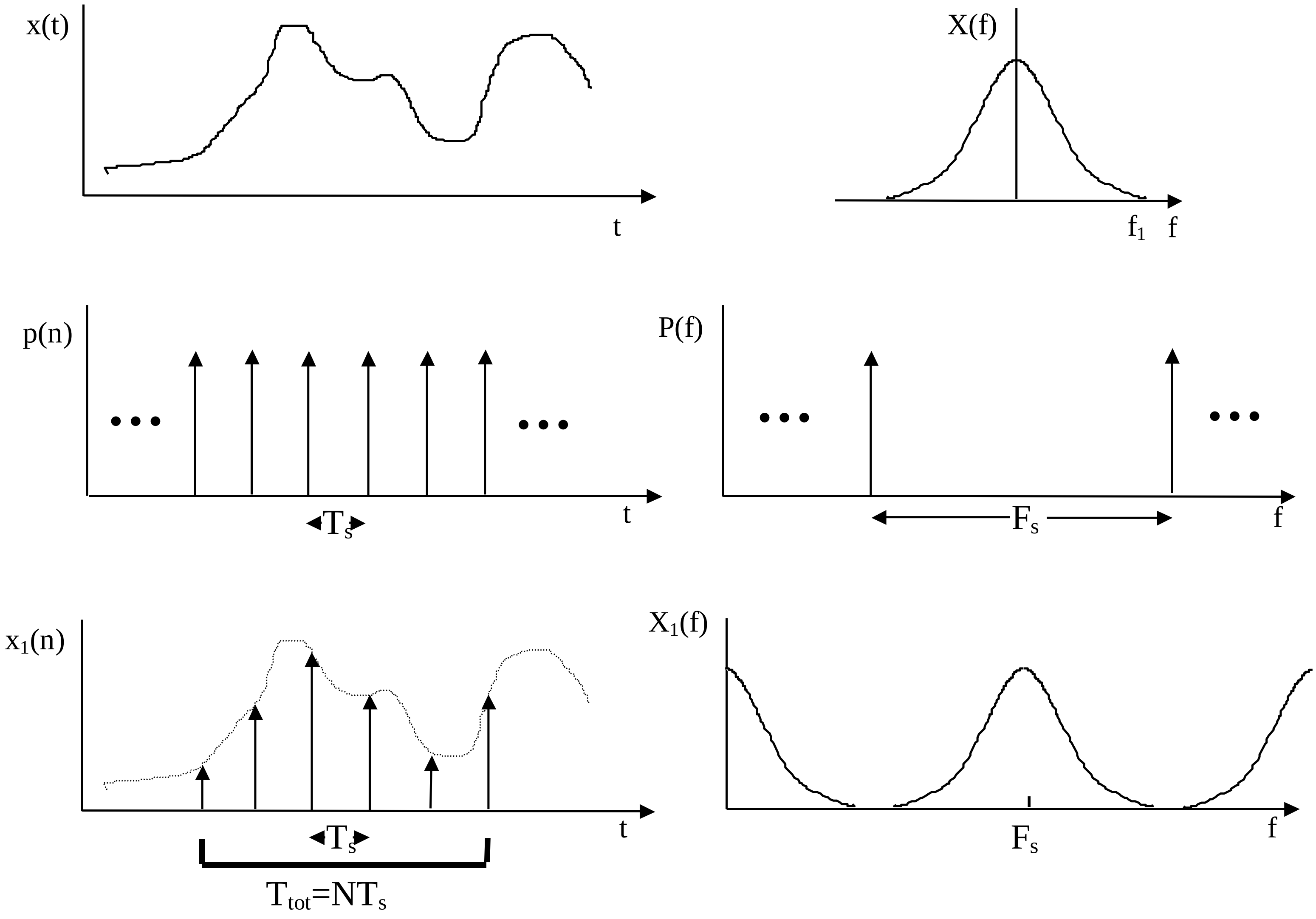
$$\omega \rightarrow k\omega_{o\lambda} = k2\pi f_{o\lambda} = k2\pi\frac{f_s}{N} = k\frac{2\pi}{T_s N} \qquad d\omega \rightarrow \omega_{o\lambda} = \frac{2\pi}{T_s N}$$

$$t \rightarrow nT_s \qquad dt \rightarrow T_s \qquad \int_{-\infty}^{\infty} \rightarrow \sum_{i=0}^{N-1}$$

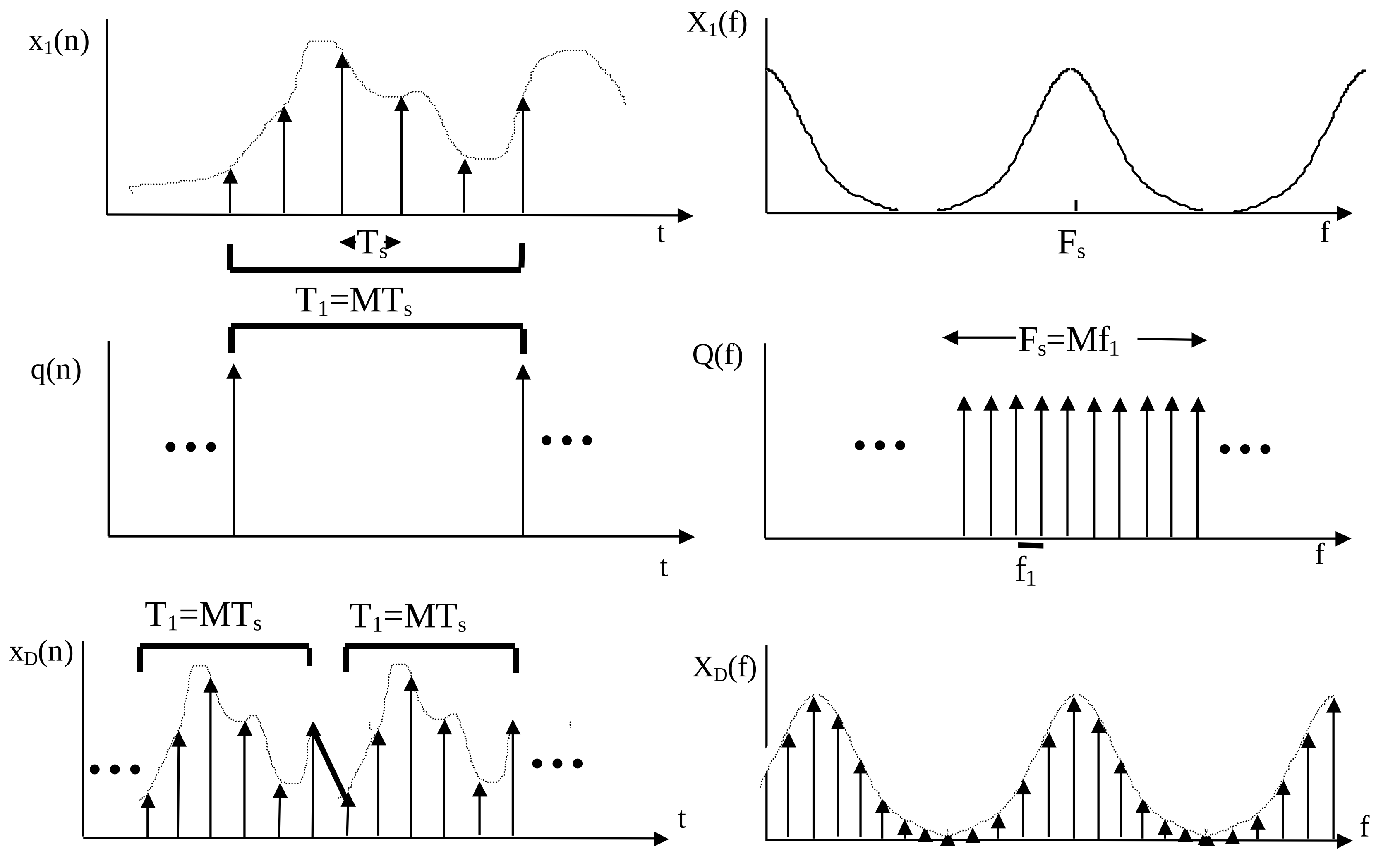
Discreet

$$X_D(k)=\sum_{n=0}^{N-1} x_D(n)\, e^{-j2\pi\frac{kn}{N}} \Leftrightarrow x_D(n)=\frac{1}{N}\sum_{k=0}^{N-1} X_D(k)\, e^{j2\pi\frac{kn}{N}} \qquad \mathrm{k,n = 0,1,\ldots N-1}$$

DFT - Graphical Derivation

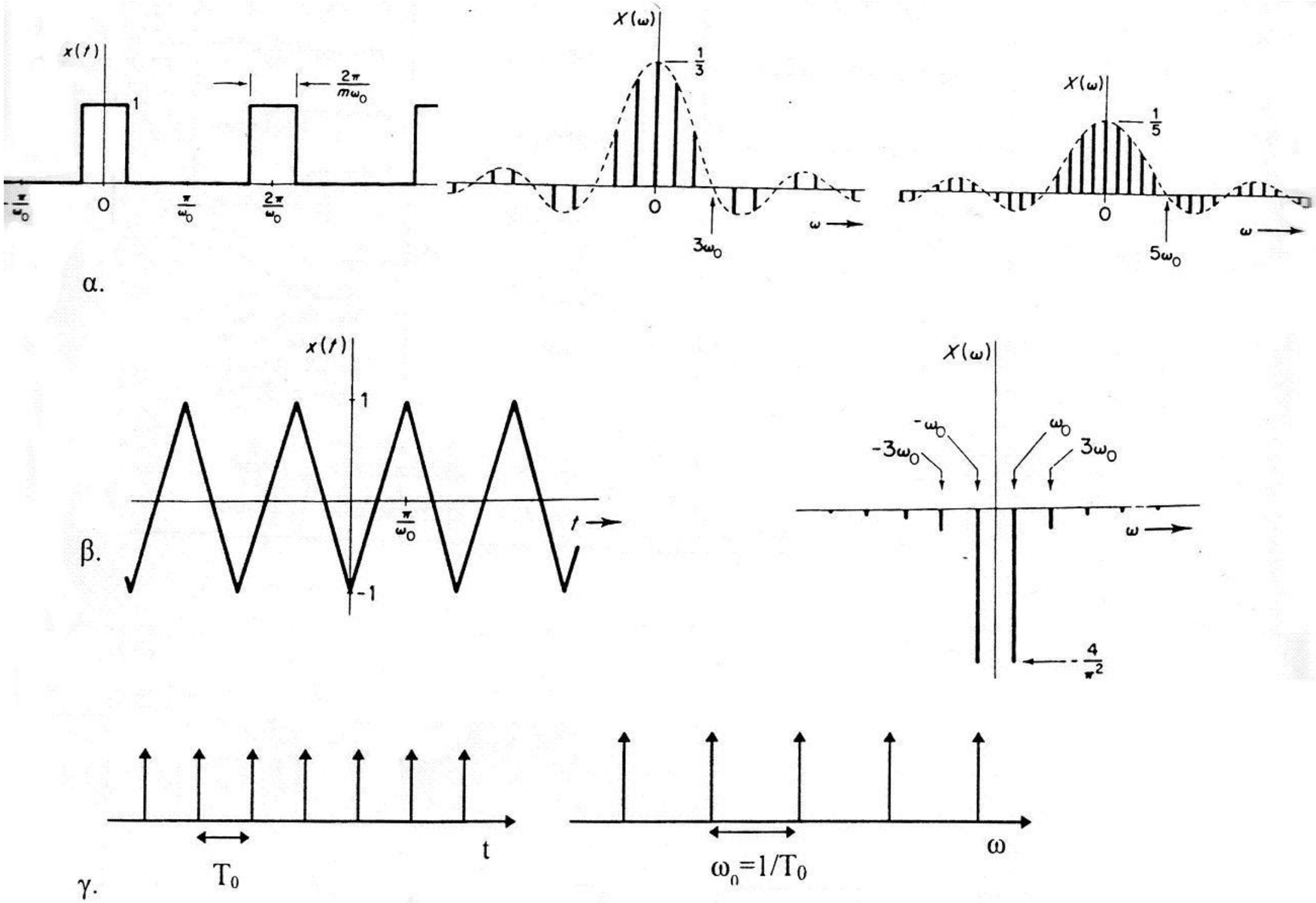


DFT - Graphical Derivation

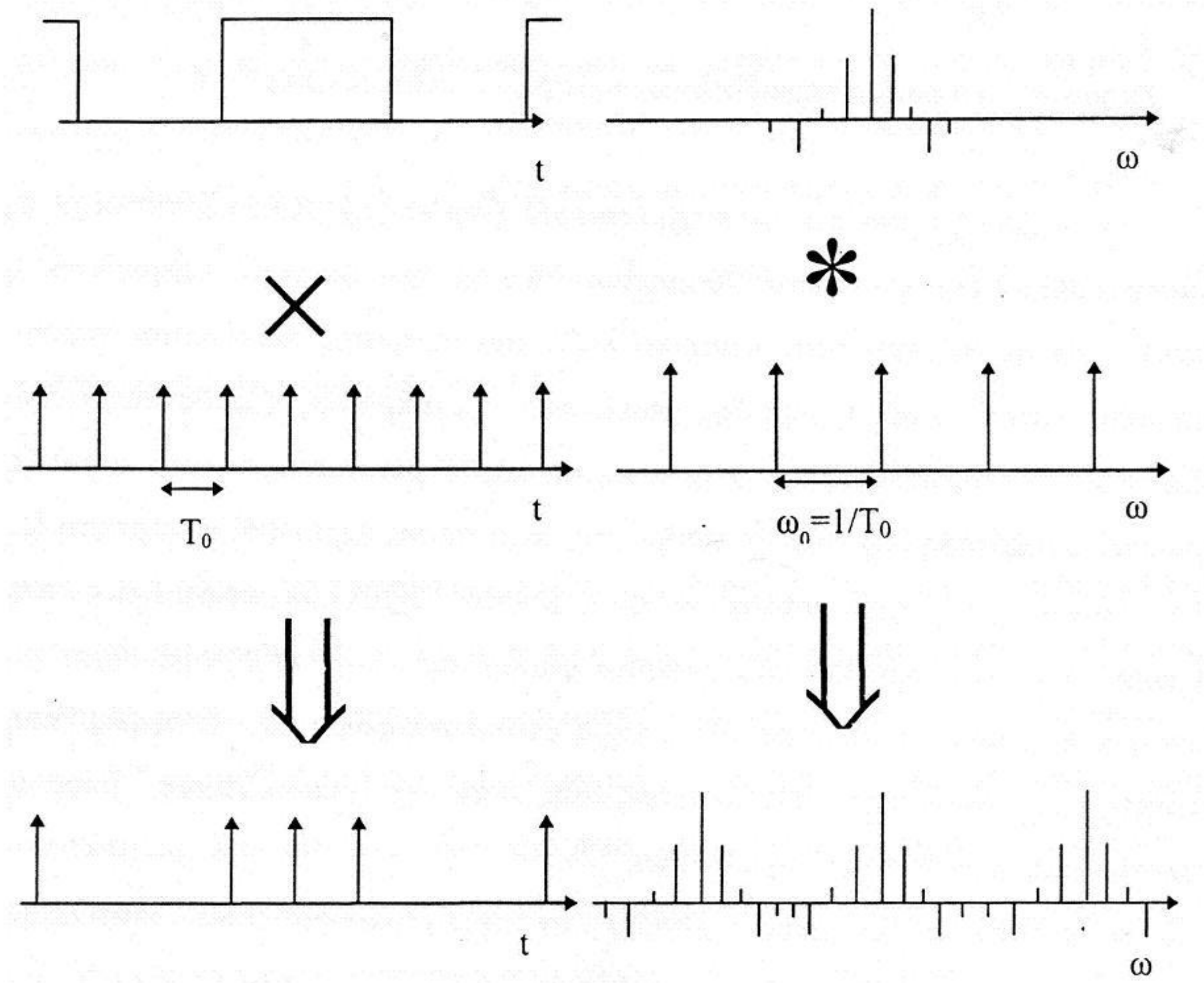


$T_1 = MT_s$ must be exactly equal to $NT_s \Rightarrow M=N$

DFT - Periodic Signals – Discreet spectrum



DFT - Periodic Signals – Discreet spectrum



DFT - Properties

1. **DFT Deals with complex quantities for x(n) and X(k).**
2. **DFT is periodic X(k)=X(k+N) thus X(N)=X(0).**
3. **DFT is symmetric |X(k)| = |X(N-k)|**
4. **Parsevals' theorem**

$$\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

5. **Convolution theorem**

$$x_3(n) = \sum_{m=0}^{N-1} x_1(m) x_2(n-m) = x_1(n) * x_2(n)$$

$$X_3(k) = X_1(k) \cdot X_2(k)$$

DFT - Example

Signal *x(n)={1,0,0,1}* n=0,1,2,3

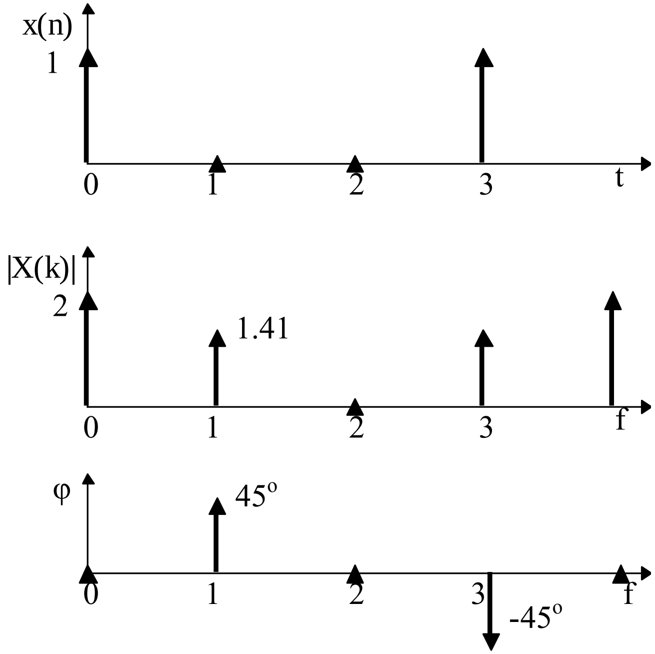
$$X(k)=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi\frac{kn}{N}} \qquad k=0,1,2,3$$

$$X(0)=\sum_{n=0}^{3} x(n)e^{-j0}=\sum_{n=0}^{3} x(n)=x(0)+x(1)+x(2)+x(3)=1+0+0+1=2$$

$$X(1)=\sum_{n=0}^{3} x(n)e^{-j2\pi n/N}=1+0+0+1e^{-j2\pi 3/4}=1+\cos(\frac{3\pi}{2})-j\sin(\frac{3\pi}{2})=1+j$$

$$X(2)=\sum_{n=0}^{3} x(n)e^{-j2\pi 2n/N}=1+0+0+1e^{-j3\pi}=1-1=0$$

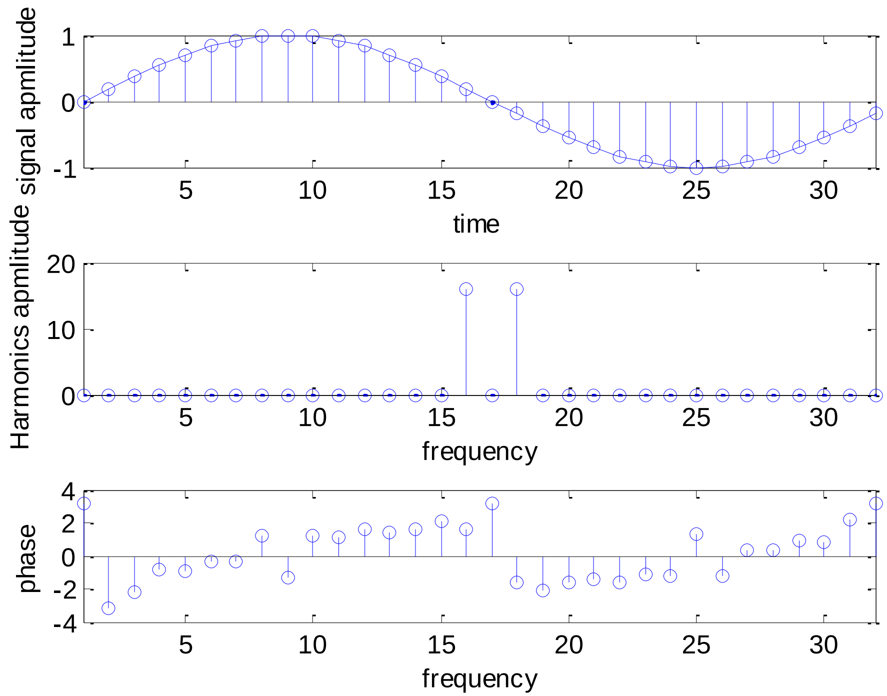
$$X(3)=\sum_{n=0}^{3} x(n)e^{-j2\pi 3n/N}=1+0+0+1e^{-j9\pi/2}=1-j$$



DFT - Example

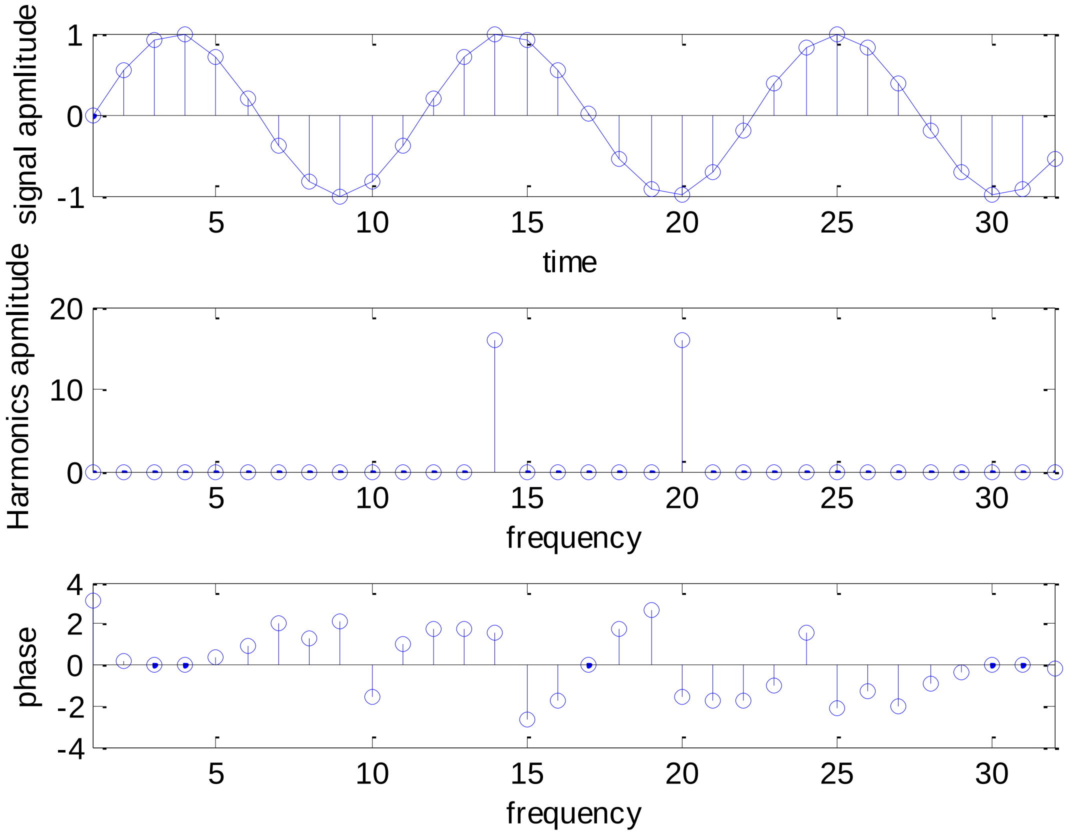
```
t=0:31;
x=sin(2*pi*t/32);
fx=fft(x);

mfx=fftshift(abs(fx));
pfx=fftshift(angle(fx));
```

$x(n)=\sin(2\pi n/32) \quad n=0,1,\ldots 31$



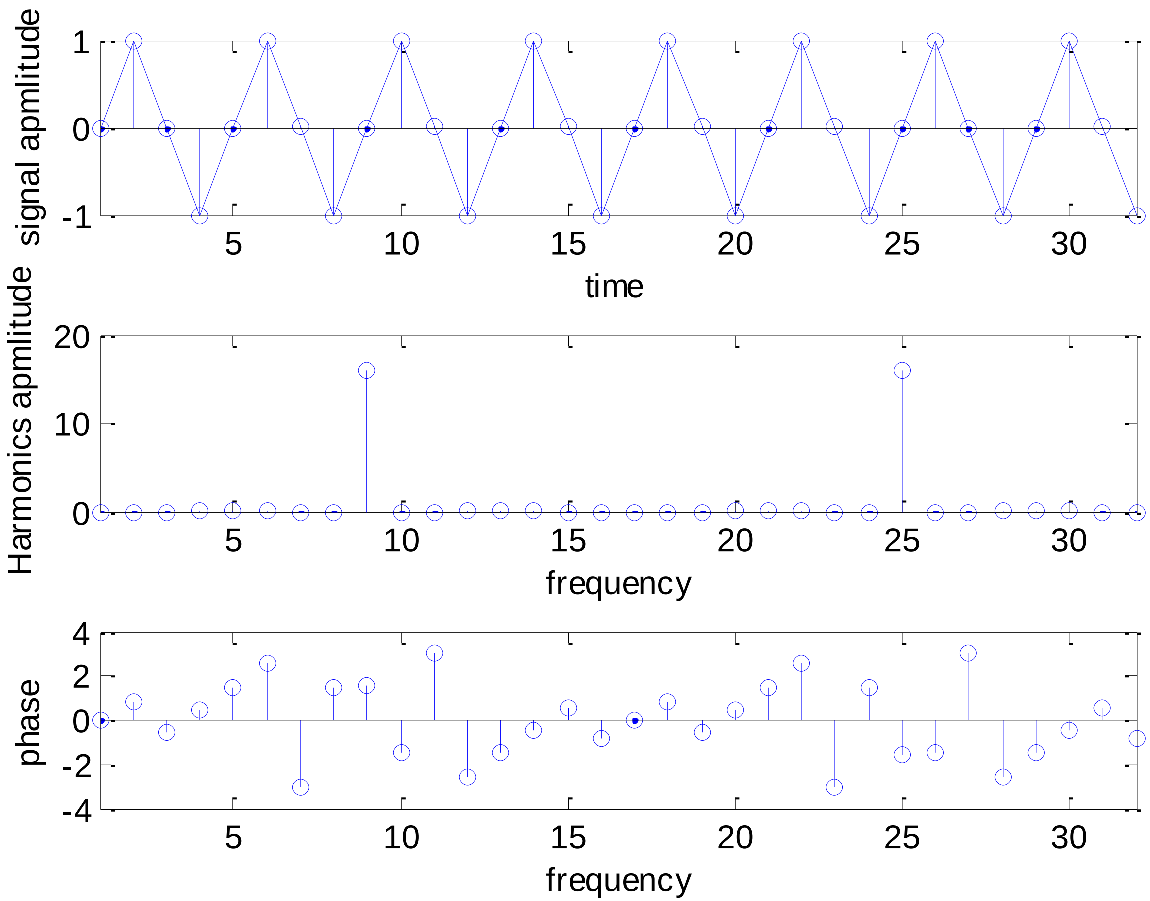
DFT - Example

$x(n)=\sin(2\pi 3n/32) \quad n=0,1,\ldots 31$



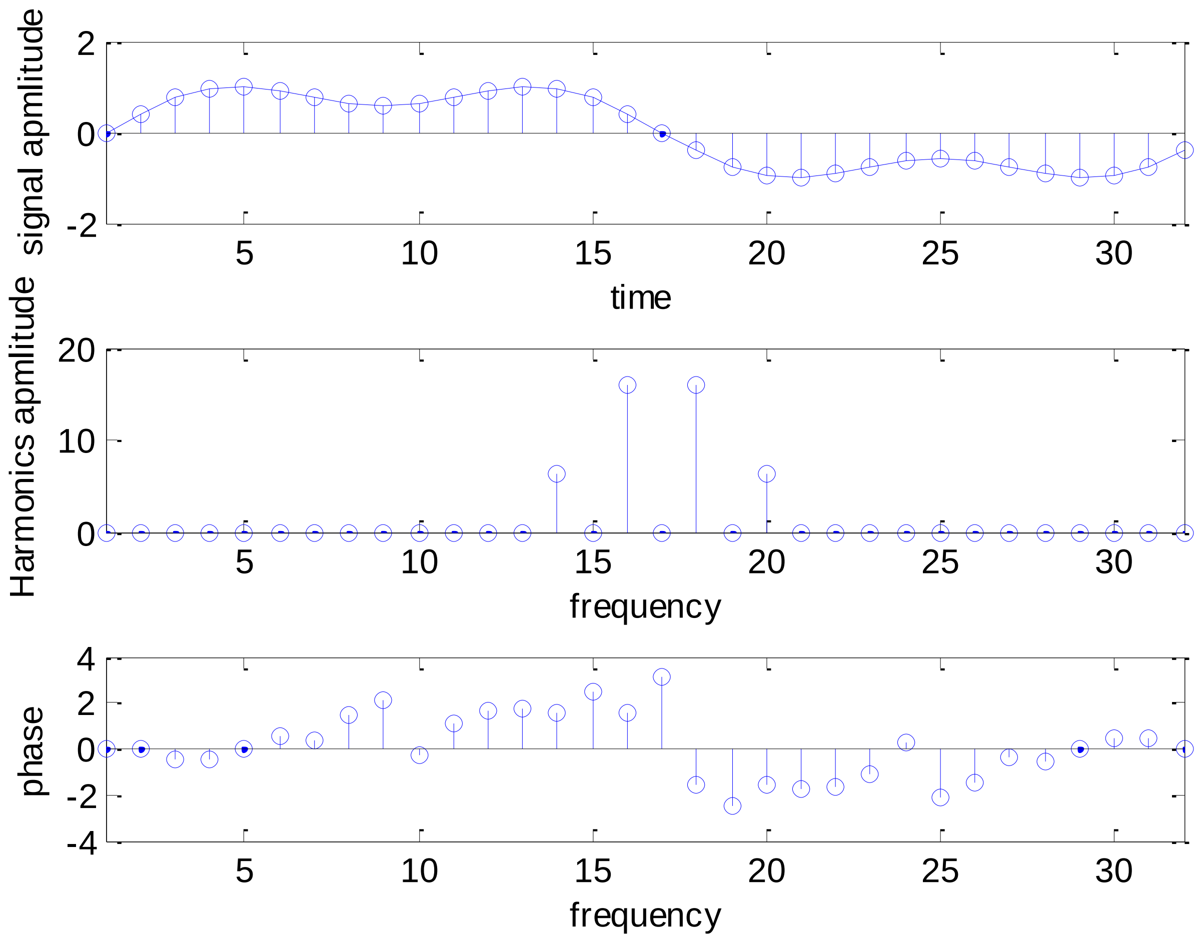
DFT - Example

$x(n)=\sin(2\pi 8n/32)$ $n=0,1,\ldots 31$



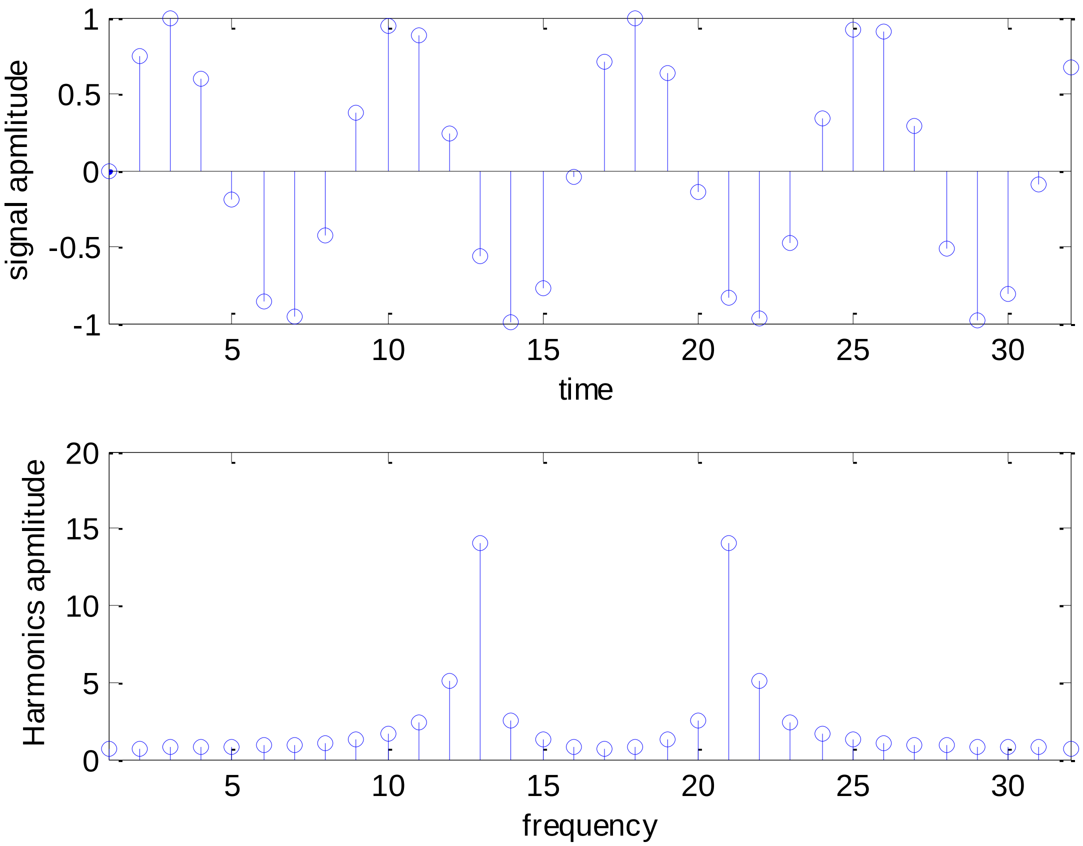
DFT - Example

$x(n)=\sin(2\pi n/32)+0.4\sin(6\pi n/32)$ $n=0,1,\ldots 31$

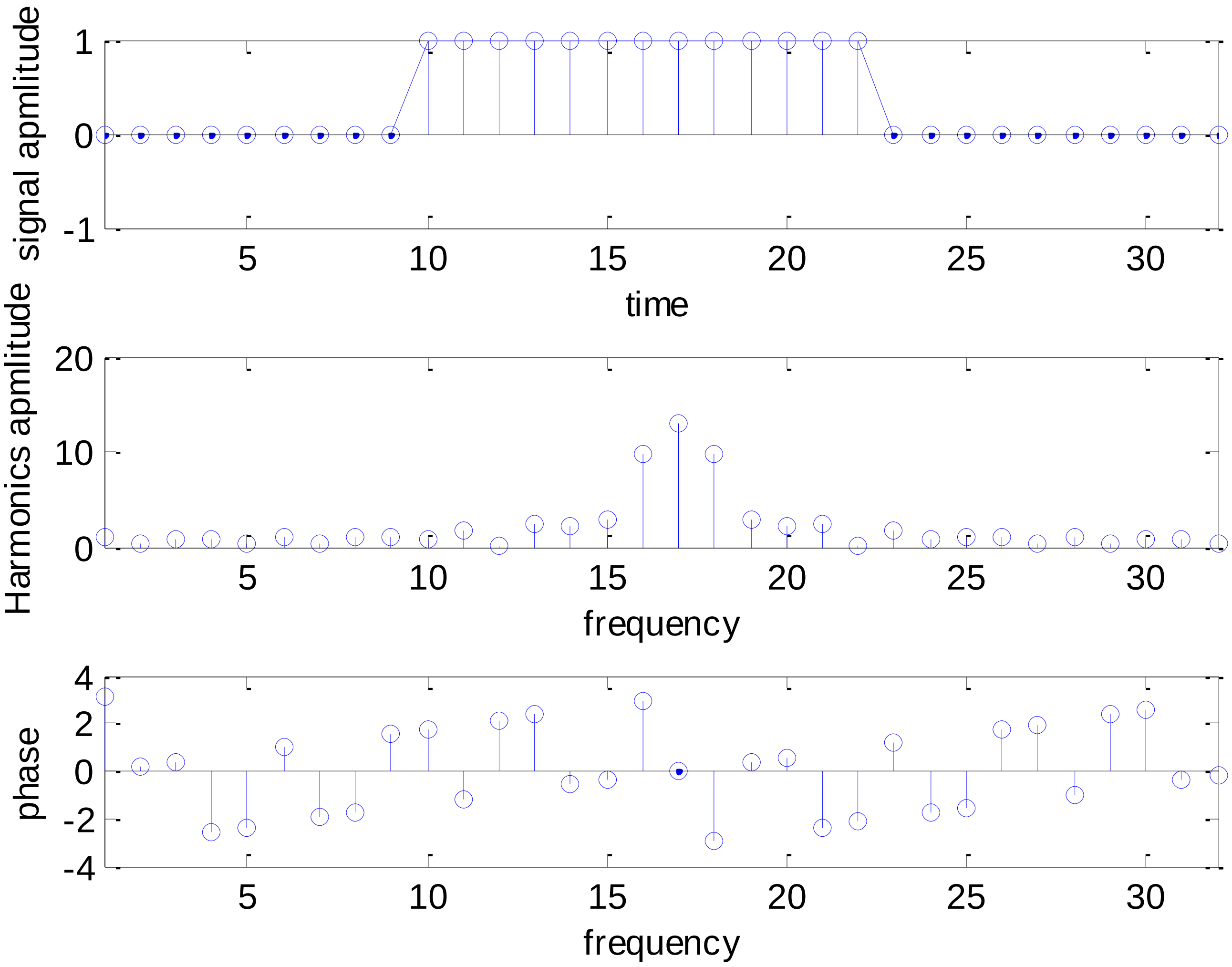


Spectral leakage

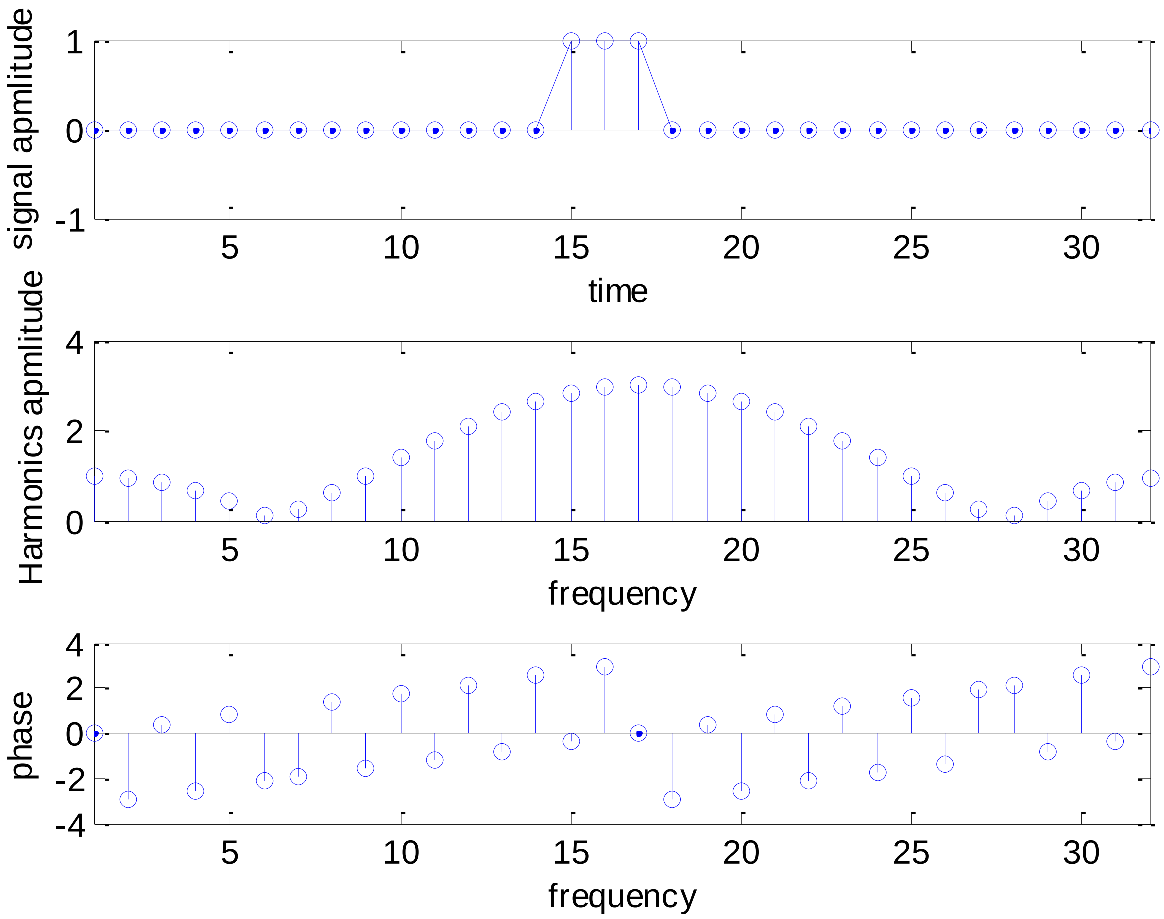
$x(n) = \sin(2\pi 4.25n/32) \quad n = 0, 1, \ldots 31$



DFT Example - Long Pulse

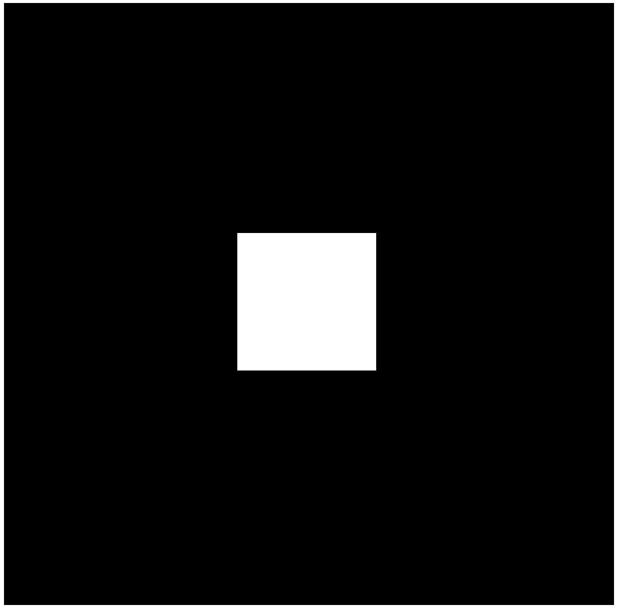


DFT Example - Short Pulse

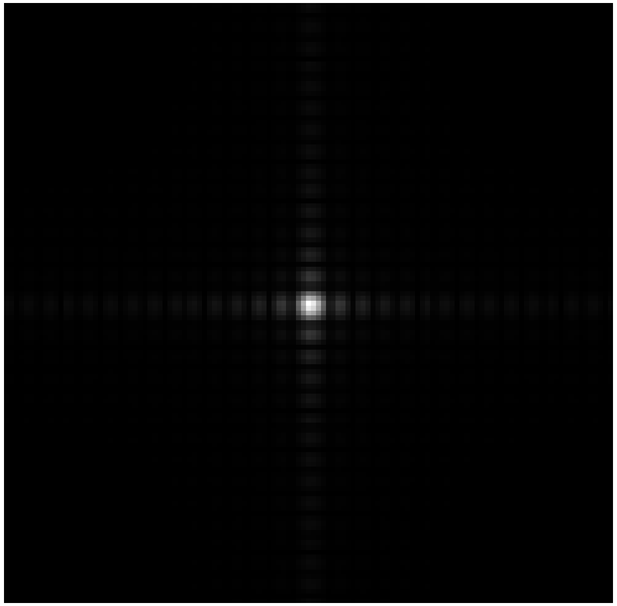


2D - DFT Example - Large Square

image

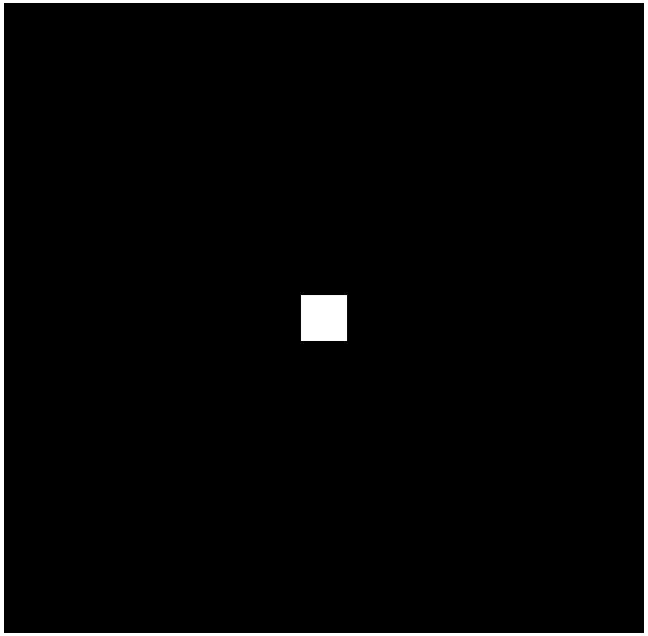


Spectral amplitude

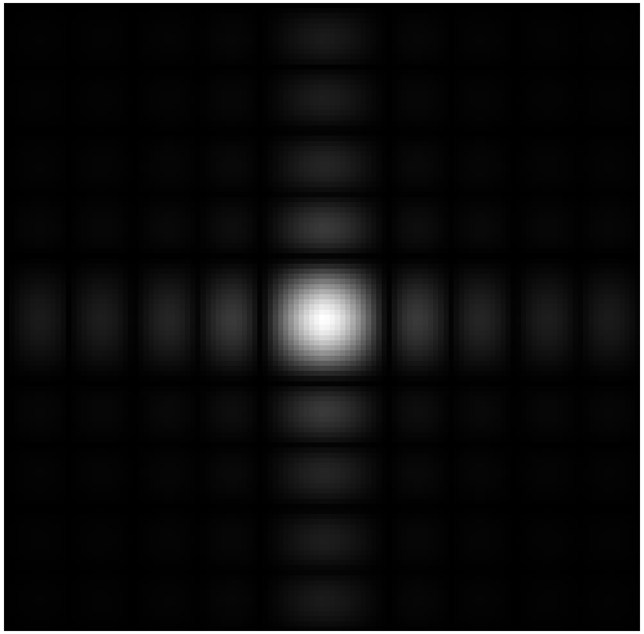


2D - DFT Example - Large Square

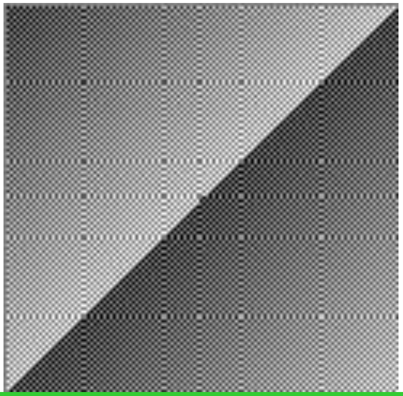
image



Spectral amplitude

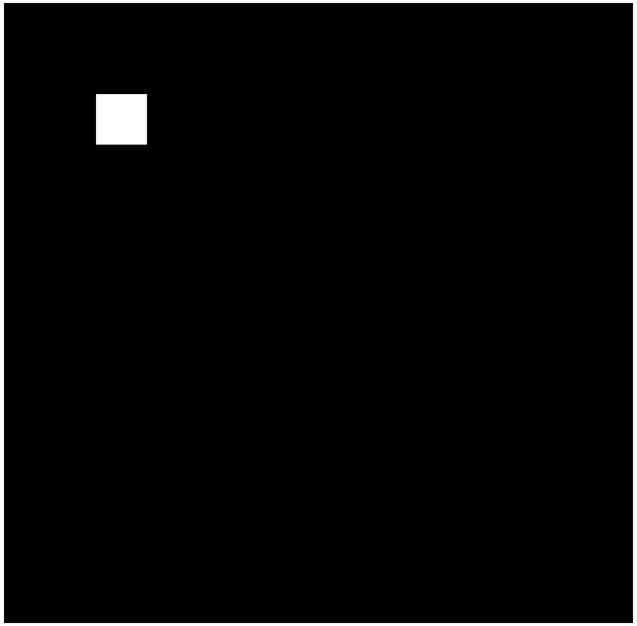


Spectral phase

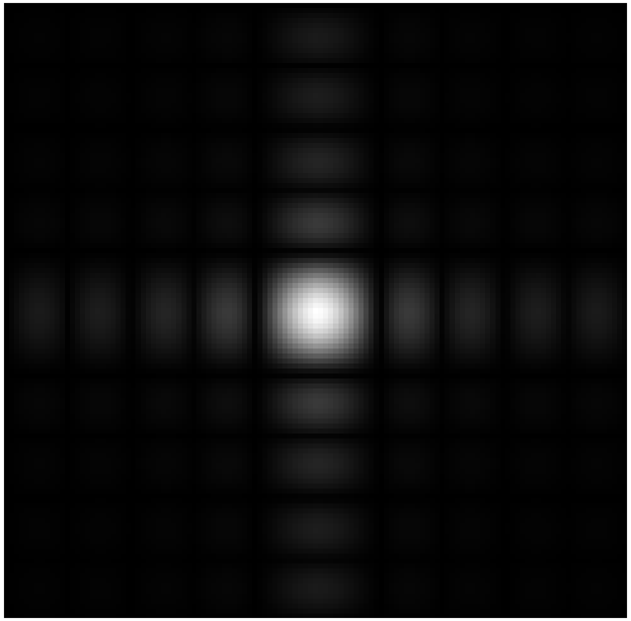


2D - DFT The importance of phase

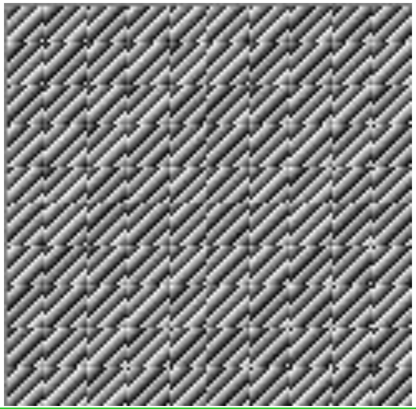
image



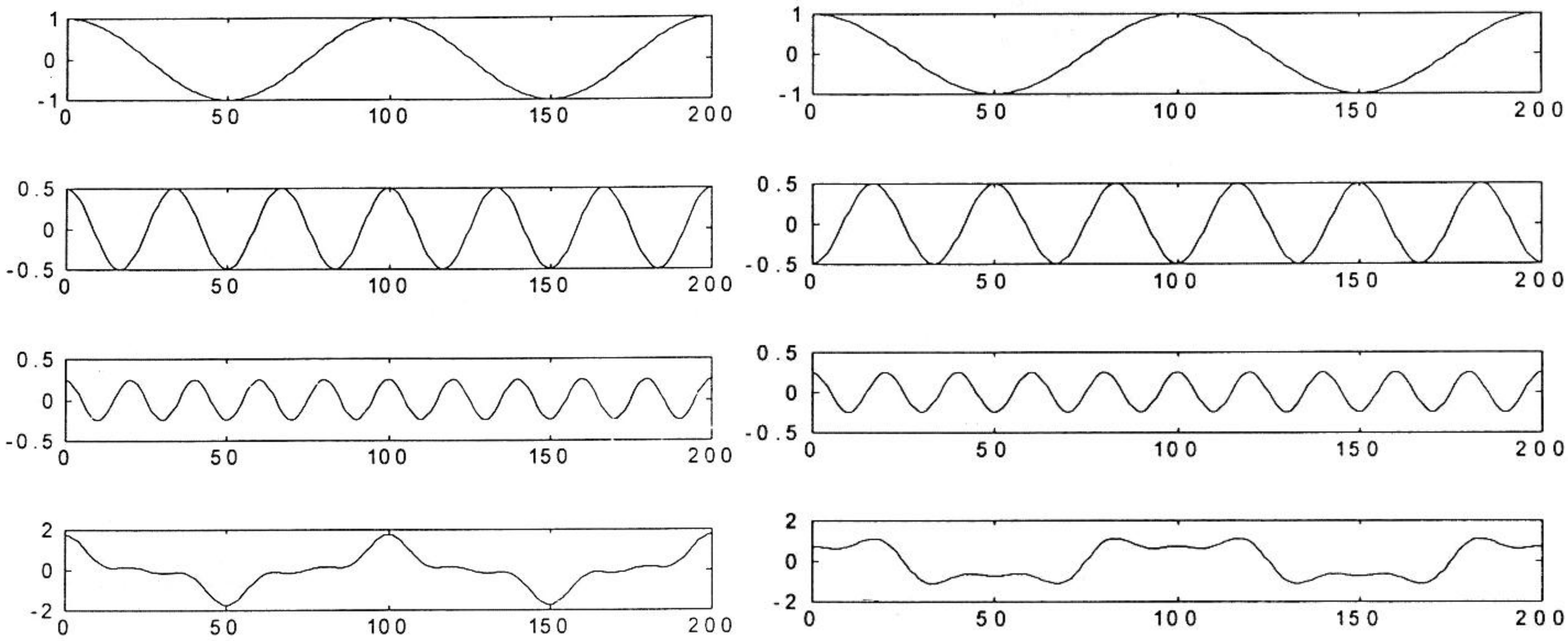
Spectral amplitude



Spectral phase

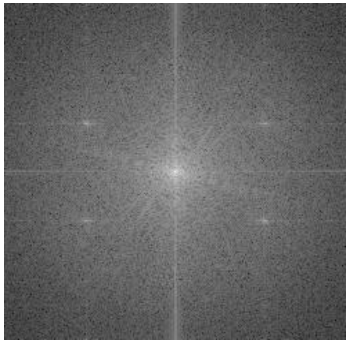


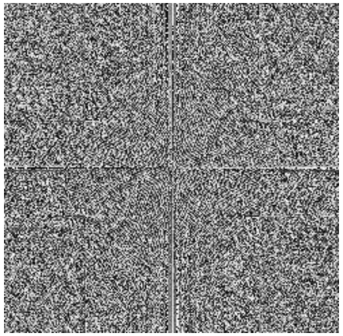
The importance of phase

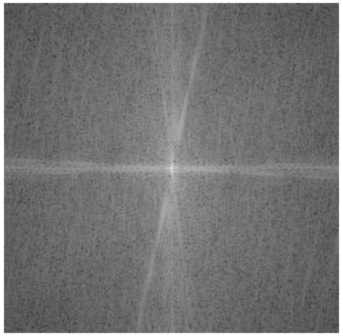


Images

Spectrum of images – amplitude and phase







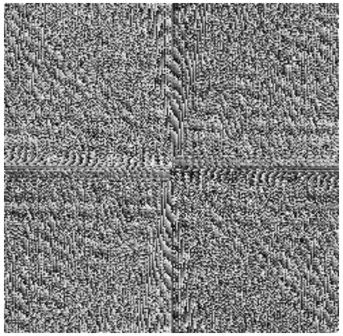
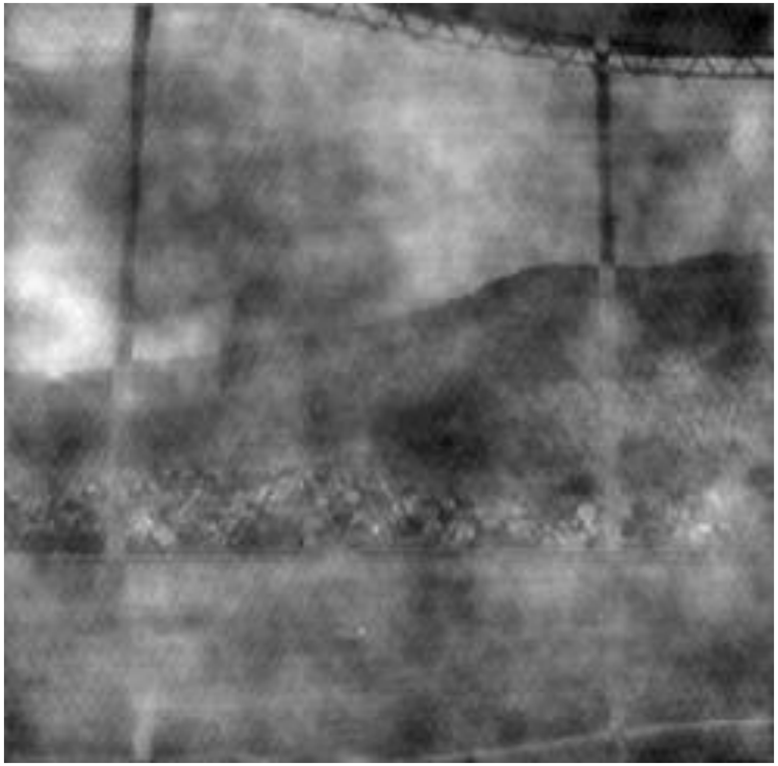


Image reconstruction from amplitude and phase

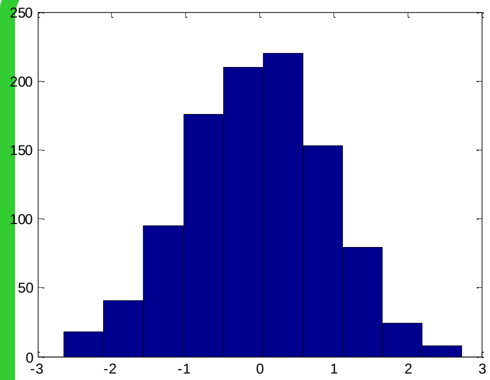
Amplitude of Vassilis
Phase of Topio



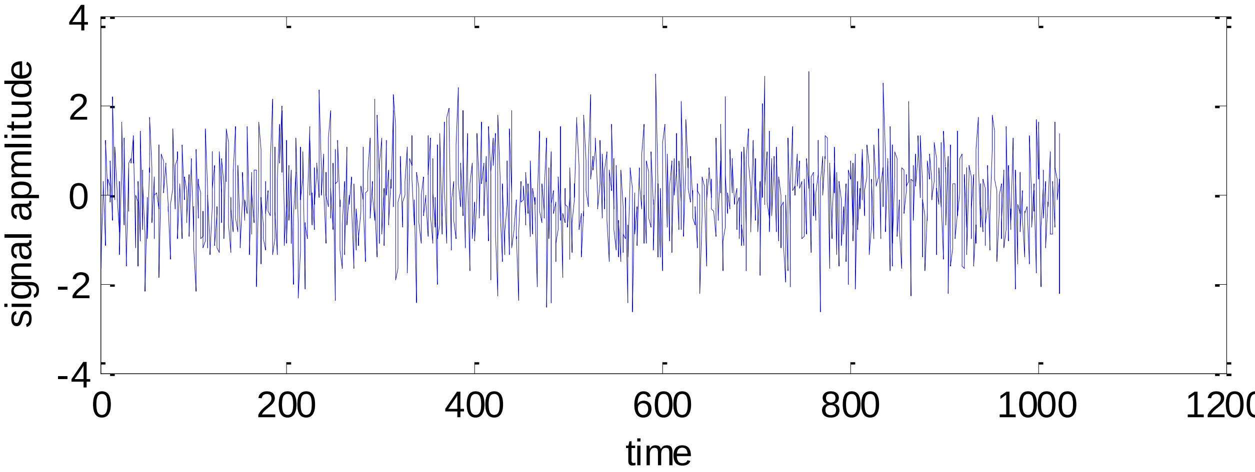
Amplitude of Topio
Phase of Vassilis

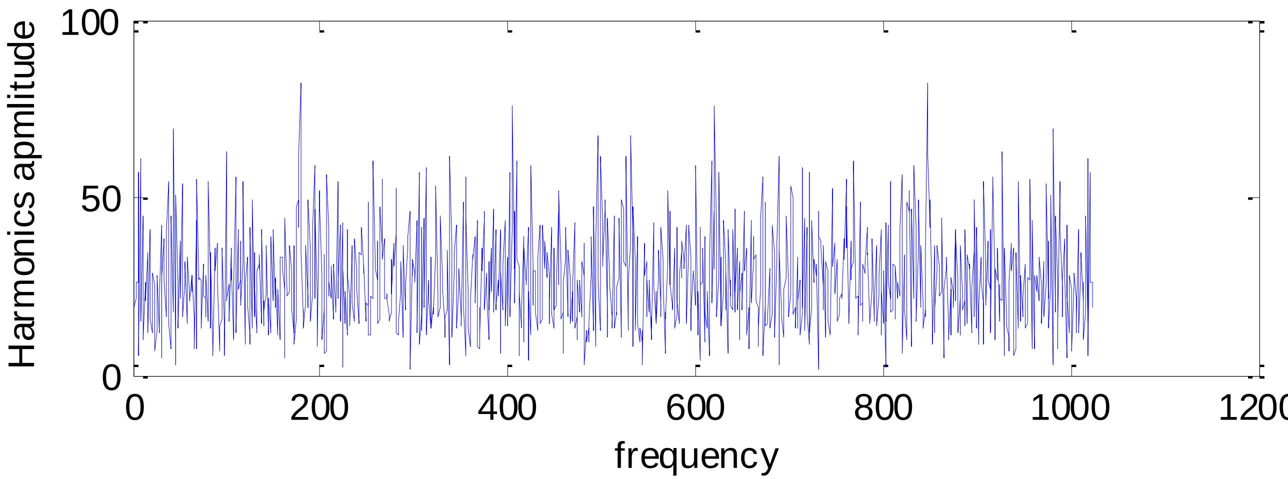
Do not change phase relationships between harmonics

White Gaussian Noise – Pdf and Spectrum

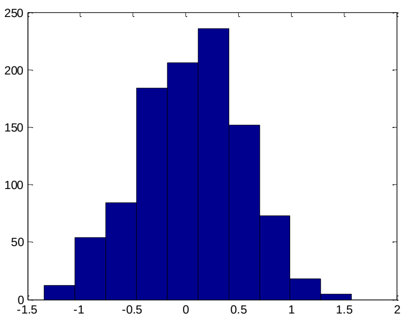


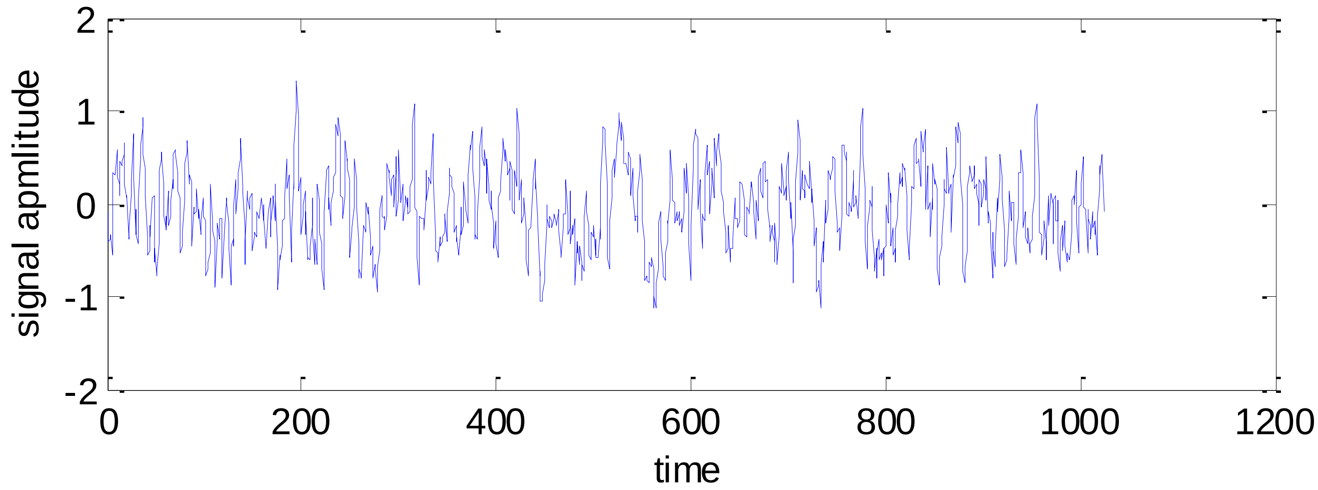
x=randn(1,1024)

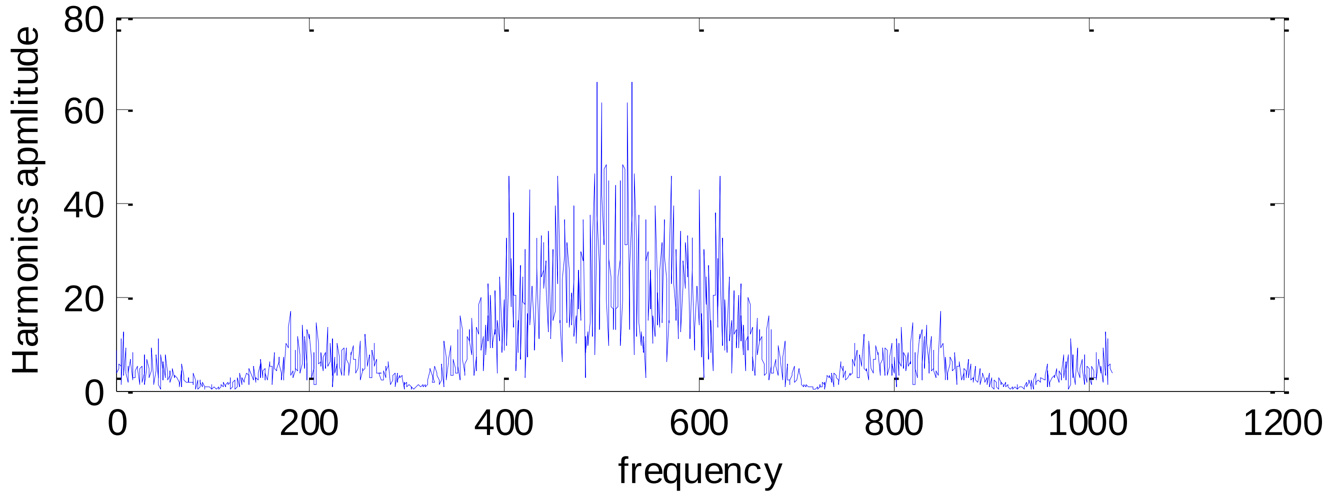




Colored Gaussian Noise – Pdf and Spectrum







DFT computational load - The FFT

The DFT $$X(k)=\sum_{n=0}^{N-1}x(n)\,e^{-j2\pi\frac{kn}{N}}\qquad k=0,1,\dots N-1$$

Evaluation of the terms $e^{-j2\pi\frac{kn}{N}}$

N^2 multiplications and N(N-1) additions

Fast Fourier Transform (FFT) (Cooley and Tukey - 1965)

- The operations are reduced by segmenting the summation in smaller parts.
- The term $e^{-j2\pi\frac{kn}{N}}$ is evaluated only once.
- Number of operations $(N/2)\log_2 N$

Basic Concept of FFT

$$X(k)=\sum_{n=0}^{N-1}x(n)\,e^{-j2\pi\frac{kn}{N}}\qquad \text{k=0,1,... N-1}$$

$$X(k)=\sum_{n=0}^{\frac{N}{2}-1}x(2n)W_N^{2nk}+\sum_{n=0}^{\frac{N}{2}-1}x(2n+1)W_N^{(2n+1)k}$$

if we put $W_N=e^{-j\frac{2\pi}{N}}$ we take $W_N^2=e^{-j2\frac{2\pi}{N}}=e^{-j\frac{2\pi}{N/2}}=W_{N/2}$

and $W_N^{(k+N/2)}=e^{-j\frac{2\pi}{N}(k+N/2)}=e^{-jk\frac{2\pi}{N}}e^{-j\pi}=-W_N^k$

so $$X(k)=\sum_{n=0}^{\frac{N}{2}-1}x(2n)W_{N/2}^{nk}+W_N^k\sum_{n=0}^{\frac{N}{2}-1}x(2n-1)W_{N/2}^{nk}$$

or $$X_1(k)=X_{21}(k)+W_N^kX_{22}(k)$$

Example – FFT of 8 points

$$X(k) = \sum_{n=0}^{7} x(n) e^{-jnk\frac{2\pi}{8}} = \sum_{n=0}^{7} x(n) W_8^{nk} =$$

$$= x(0)W_8^0 + x(1)W_8^k + x(2)W_8^{2k} + x(3)W_8^{3k} + x(4)W_8^{4k} + x(5)W_8^{5k} + x(6)W_8^{6k} + x(7)W_8^{7k} =$$

$$= \left[x(0) + x(2)W_8^{2k} + x(4)W_8^{4k} + x(6)W_8^{6k}\right] + \mathrm{W}_8^{\mathrm{k}} \left[x(1) + x(3)W_8^{2k} + x(5)W_8^{4k} + x(7)W_8^{6k}\right]$$

and since $W_8^{2k} = W_4^k$ we have

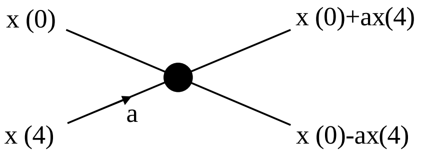
$$X(k) = \left[x(0) + x(2)W_4^k + x(4)W_4^{2k} + x(6)W_4^{3k}\right] + \mathrm{W}_8^{\mathrm{k}} \left[x(1) + x(3)W_4^k + x(5)W_4^{2k} + x(7)W_4^{3k}\right]$$

In the same way

$$X(k) = \left\{\left[x(0) + x(4)W_2^k\right] + \mathrm{W}_4^{\mathrm{k}}\left[x(2) + x(6)W_2^k\right]\right\} +$$

$$\mathrm{W}_8^{\mathrm{k}} \left\{\left[x(1) + x(5)W_2^k\right] + W_4^k\left[x(3) + x(7)W_2^k\right]\right\}$$

Number of points → power of 2



Simplified evaluation of the 8 harmonics

$$X(0)=\{[x(0)+x(4)]+[x(2)+x(6)]\}+\{[x(1)+x(5)]+[x(3)+x(7)]\}$$

$$X(1)=\{[x(0)-x(4)]+W_8^2[x(2)-x(6)]\}+W_8^1\{[x(1)-x(5)]+W_8^2[x(3)-x(7)]\}$$

$$X(2)=\{[x(0)+x(4)]-[x(2)+x(6)]\}+W_8^2\{[x(1)+x(5)]-[x(3)+x(7)]\}$$

$$X(3)=\{[x(0)-x(4)]-W_8^2[x(2)-x(6)]\}+W_8^3\{[x(1)-x(5)]-W_8^2[x(3)-x(7)]\}$$

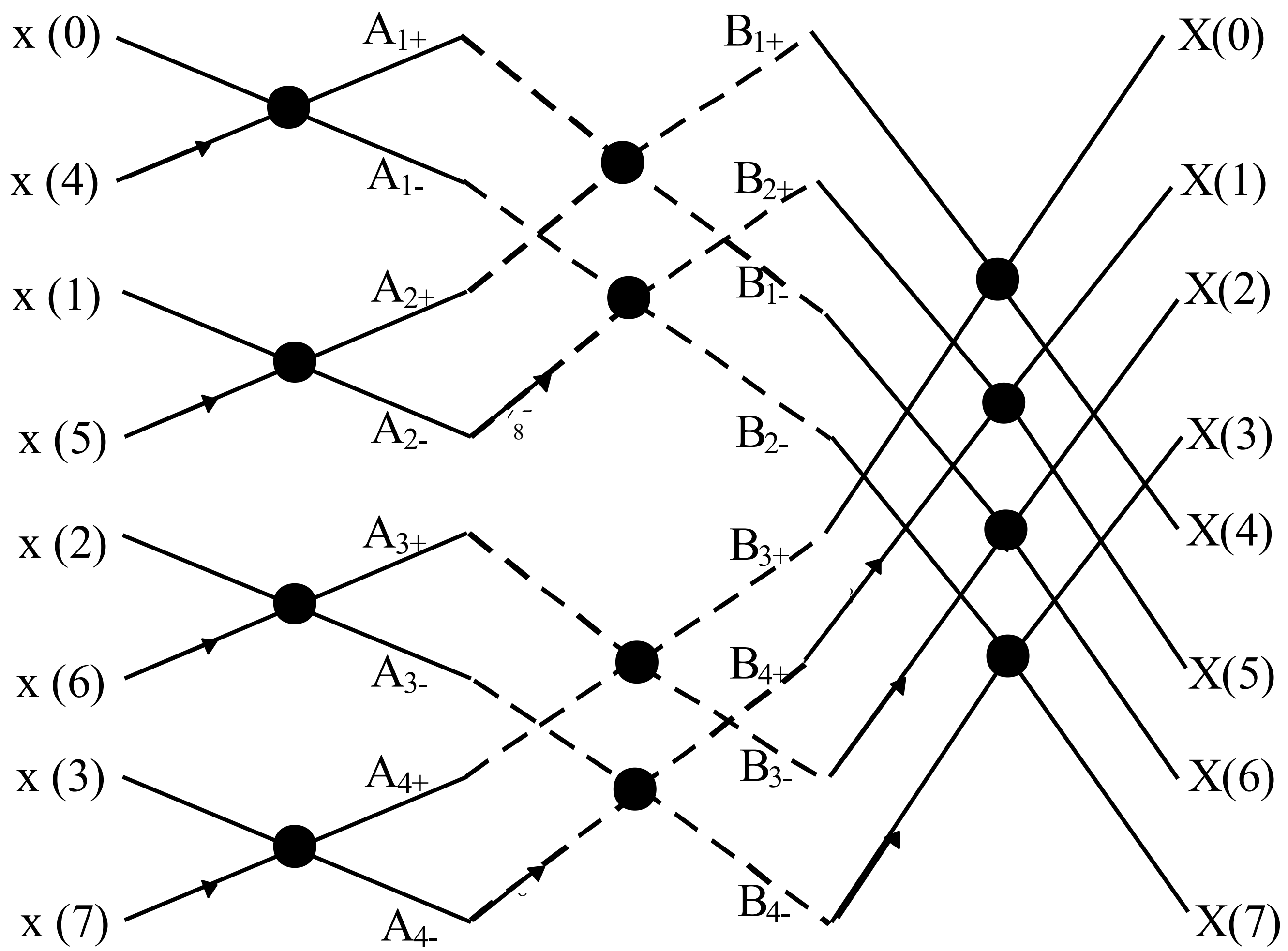
$$X(4)=\{[x(0)+x(4)]+[x(2)+x(6)]\}-\{[x(1)+x(5)]+[x(3)+x(7)]\}$$

$$X(5)=\{[x(0)-x(4)]+W_8^2[x(2)-x(6)]\}-W_8^1\{[x(1)-x(5)]+W_8^2[x(3)-x(7)]\}$$

$$X(6)=\{[x(0)+x(4)]-[x(2)+x(6)]\}-W_8^2\{[x(1)+x(5)]-[x(3)+x(7)]\}$$

$$X(7)=\{[x(0)-x(4)]-W_8^2[x(2)-x(6)]\}-W_8^3\{[x(1)-x(5)]-W_8^2[x(3)-x(7)]\}$$

The butterfly configuration



The END

Do not drink

too much coffee

Back on Monday