Basics on Digital Signal Processing

z - transform - Digital Filters

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Outline of the Lecture

1. The z-transform
2. Properties
3. Examples
4. Digital filters
5. Design of IIR and FIR filters
6. Linear phase

z-Transform

$$X(k)=\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}nk}$$

Time to frequency

$$X(z)=\sum_{n=0}^{N-1} x(n)\, z^{-n}$$

Time to z-domain (complex)

Transformation tool is the complex wave

$$e^{j\omega}=e^{j2\pi k/N}$$

With amplitude $|e^{j\omega}|=1$

Transformation tool is

$$z=\rho\, e^{j\omega}=e^{j\omega+\sigma}$$

With amplitude ρ changing(?) with time

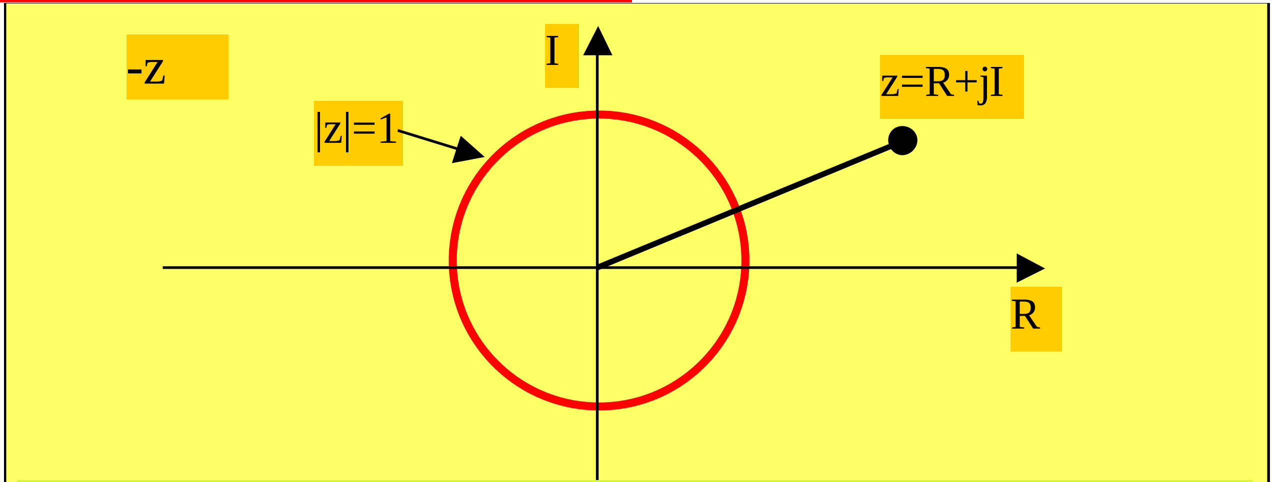
The z-transform is more general than the DFT

z-Transform

For $z=e^{j\omega}$ i.e. $\rho=1$ we work on the unit circle

And the z-transform degenerates into the Fourier transform.

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n}$$

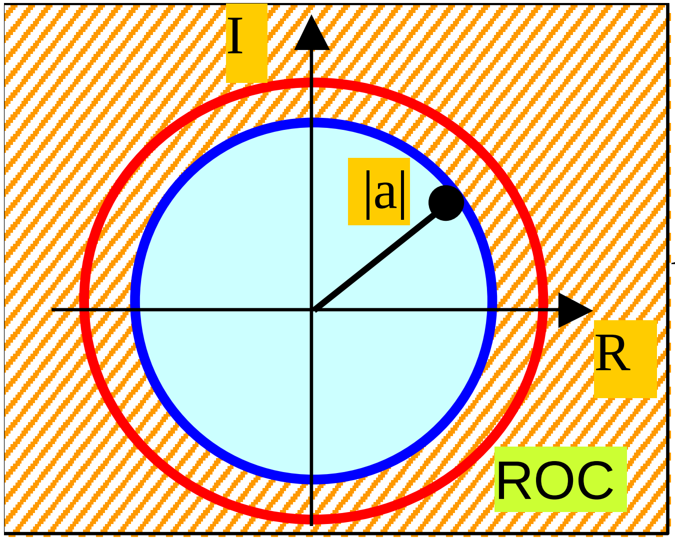


The DFT is an expression of the z-transform on the unit circle.

The quantity X(z) must exist with finite value on the unit circle i.e. must posses spectrum with which we can describe a signal or a system.

z-Transform convergence

We are interested in those values of z for which X(z) converges.
This region should contain the unit circle.



Why is it so?

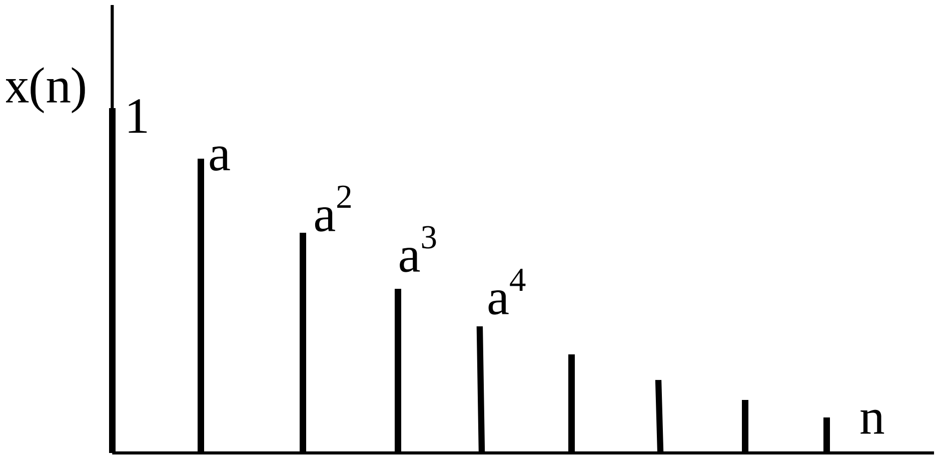
$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n} = \sum_{n=0}^{N-1} x(n)\,(1/z^{n})$$

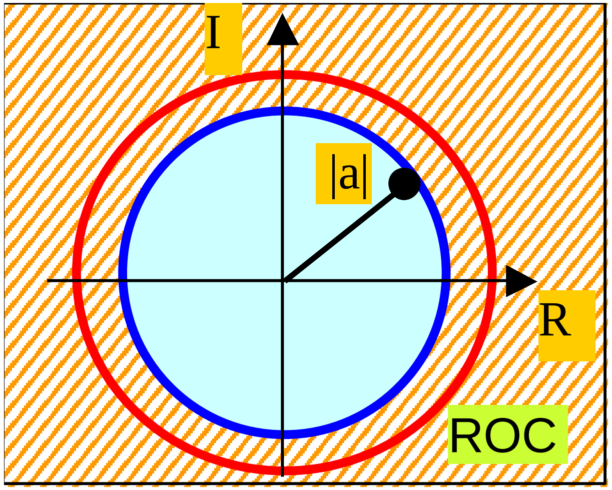
At z=0, X(z) diverges

The values of z for which X(z) diverges are called poles of X(z).

z-Transform example

Which is the z-transform and the ROC of a discrete time sequence
$x(n)=a^n$ for $n \geq 0$ and $a<1$?





$$X(z)=\sum_{n=0}^{N-1} x(n)\, z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

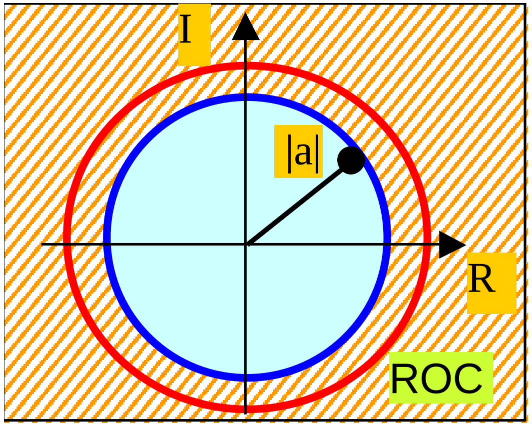
which for $|az^{-1}|<1$ or $|z|>|a|$ converges to

$$X(z)=\frac{1}{1-az^{-1}}=\frac{z}{z-a}$$

The pole z=a, is never included in the ROC

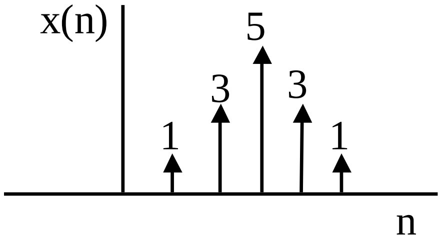
Poles and zeros of X(z)

Poles: X(z)=∞ Zeros: X(z)=0



For infinite sequences, X(z) converges everywhere outside the circle with radius the pole with maximum value.

For finite sequences, X(z) converges everywhere except at z=0



$$X(z)=\sum_{n=-\infty}^{\infty} x(n)z^{-n}=z^{-1}+3z^{-2}+5z^{-3}+3z^{-4}+z^{-5}$$

For stable, causal digital systems the region of convergence includes the Unit Circle so that the system possesses spectrum

z-Transform general form

$$X(z)=\frac{a_0+a_1z^{-1}+a_2z^{-2}+\cdots+a_Nz^{-N}}{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_Nz^{-M}}$$

$$X(z)=\frac{k(z-z_1)(z-z_2)\cdots(z-z_N)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

Those values of z *(z_i)* that make the nominator zero are called zeros.

While the poles are the values of z *(p_i)* that make the denominator zero and thus X(z) diverges.

For stable, causal digital systems the region of convergence includes the Unit Circle so that the system possesses spectrum

$$H(e^{j\omega T})=H(z)\Big|_{z=e^{j\omega T}}=\sum_{n=-\infty}^{\infty}h(n)z^{-n}\Bigg|_{z=e^{j\omega T}}=\sum_{n=-\infty}^{\infty}h(n)e^{-j\omega T}$$

Inverse z-Transform

A simple way to evaluate the signal from the X(z) is to perform the division

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0.356z^{-2}}$$

$$X(z) = 1 + 3z^{-1} + 3.6439z^{-2} + 2.5756z^{-3} + \cdots$$

The signal is x(0)=1, x(1)=3, x(2)=3.6439, x(3)=2.5756

z-Transform properties

Linearity

$$x_1(n) \leftrightarrow X_1(z)$$
$$x_2(n) \leftrightarrow X_2(z)$$
$$\Rightarrow ax_1(n) + bx_2(n) \leftrightarrow aX_1(z) + bX_2(z)$$

Delay or Shift

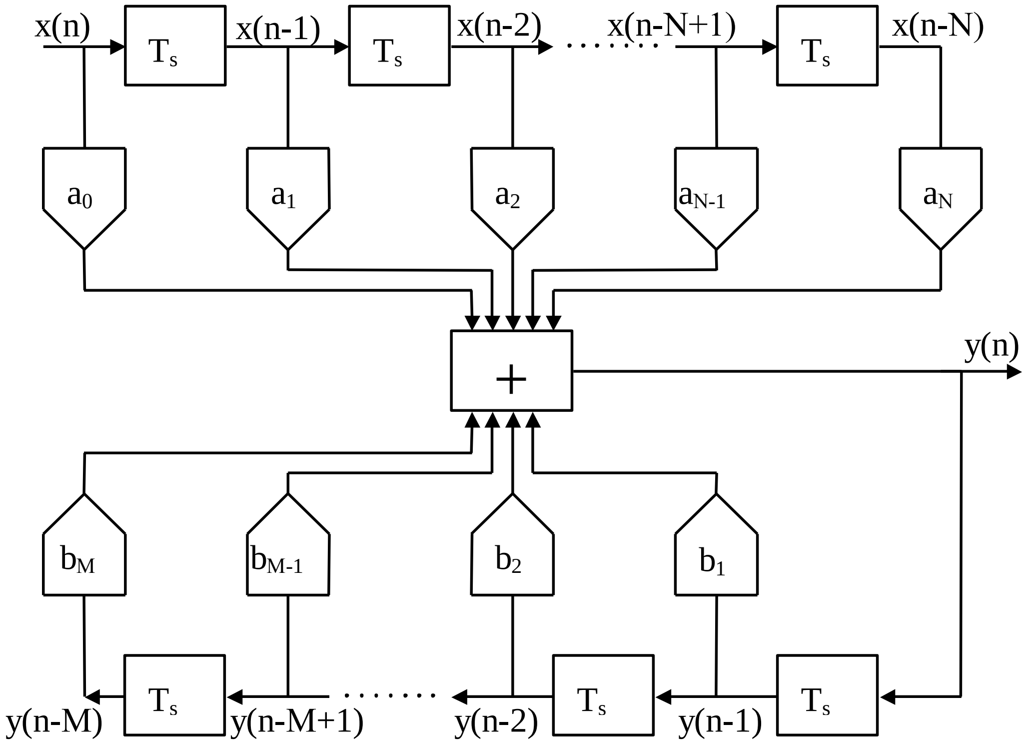
$$x(n) \leftrightarrow X(z)$$
$$\Rightarrow x(n-m) \leftrightarrow z^{-m} X(z)$$

Convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \Leftrightarrow Y(z) = H(z)X(z)$$

z-Transform and digital systems

$$y(n)=\sum_{k=0}^{N}a_k x(n-k)-\sum_{k=1}^{M}b_k y(n-k)$$



$$H(z)=\frac{\sum_{k=0}^{N}a_k z^{-k}}{1+\sum_{k=1}^{N}b_k z^{-k}}$$

Digital system z-Transform derivation

$$y(n)=\sum_{k=0}^{N}a_k x(n-k)-\sum_{k=1}^{M}b_k y(n-k)$$

z-transform both sides

$$\sum_{n=0}^{\infty}z^{-n}y(n)=\sum_{n=0}^{\infty}z^{-n}\sum_{k=0}^{N}a_k x(n-k)-\sum_{n=0}^{\infty}z^{-n}\sum_{k=1}^{M}b_k y(n-k)$$

Interchange summations

$$Y(z)=\sum_{k=0}^{N}a_k\sum_{n=0}^{\infty}z^{-n}x(n-k)-\sum_{k=1}^{M}b_k\sum_{n=0}^{\infty}z^{-n}y(n-k)$$

Replace transforms

$$Y(z)=\sum_{k=0}^{N}a_k X(z-k)-\sum_{k=1}^{M}b_k Y(z-k)$$

Shift property

$$Y(z)=\sum_{k=0}^{N}a_k z^{-k}X(z)-\sum_{k=1}^{M}b_k z^{-k}Y(z)$$

Common factors

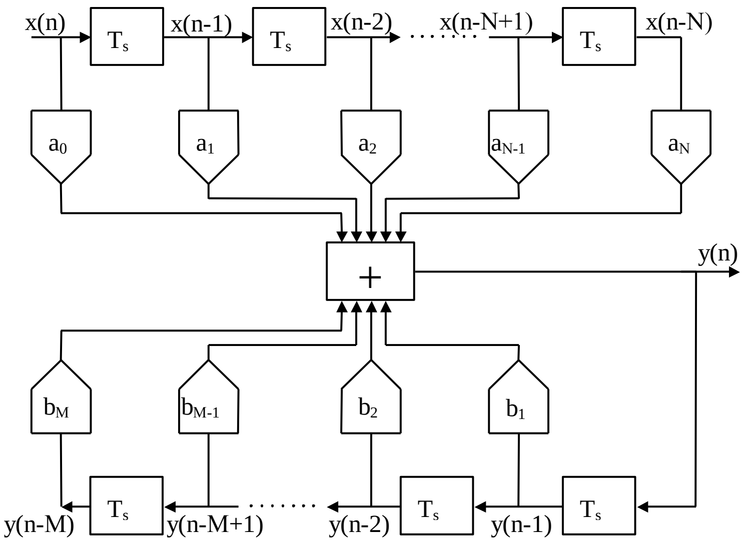
$$Y(z)=X(z)\sum_{k=0}^{N}a_k z^{-k}-Y(z)\sum_{k=1}^{M}b_k z^{-k}$$

Digital system z-Transform derivation

$$Y(z)+Y(z)\sum_{k=1}^{M} b_k z^{-k} = X(z)\sum_{k=0}^{N} a_k z^{-k}$$

Common factors

$$Y(z)(1+\sum_{k=1}^{M} b_k z^{-k}) = X(z)\sum_{k=0}^{N} a_k z^{-k}$$



$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1+\sum_{k=1}^{N} b_k z^{-k}} = H(z)$$

Frequency

$$y(n) = \sum_{k=0}^{N} a_k x(n-k) - \sum_{k=1}^{M} b_k y(n-k)$$

Time

Application

Find the impulse response h(n) and the input-output relationship of the filter described by

$$H(z)=\frac{1-z^{-1}}{1+0.5z^{-1}}$$

Examine the stability of the filter and find its frequency response.

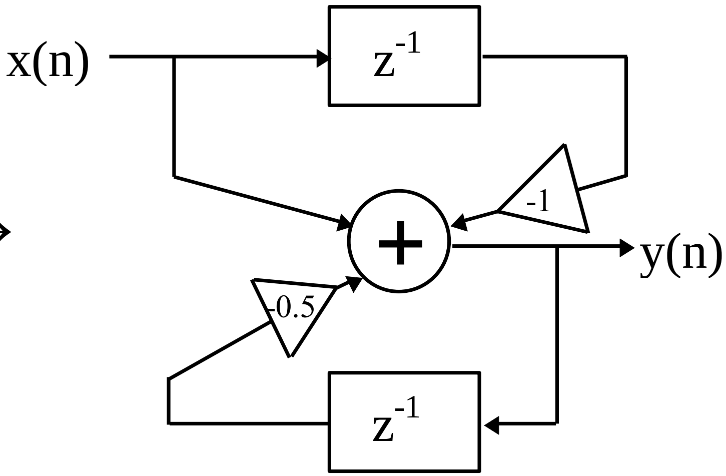
Solution

$$H(z)=\frac{Y(z)}{X(z)}=\frac{1-z^{-1}}{1+0.5z^{-1}}\Rightarrow$$

$$Y(z)+0.5Y(z)z^{-1}=X(z)-X(z)z^{-1}\xrightarrow{Z^{-1}}$$

$$y(n)+0.5y(n-1)=x(n)-x(n-1)\Rightarrow$$

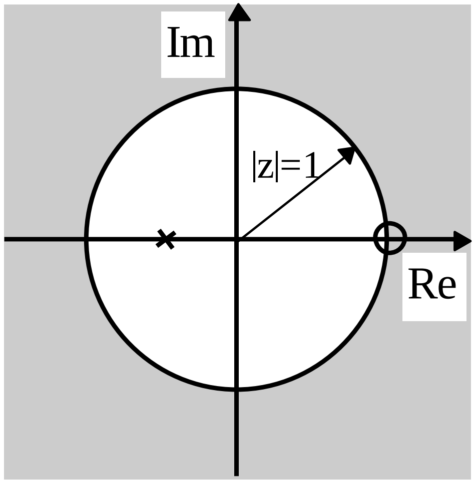
$$y(n)=x(n)-x(n-1)-0.5y(n-1)$$



h(n) is obtained from the terms of the polynomial which results after the division

$$(1-z^{-1})/(1+0.5z^{-1})=1-1.5z^{-1}+0.75z^{-2}-0.375z^{-3}+\cdots$$

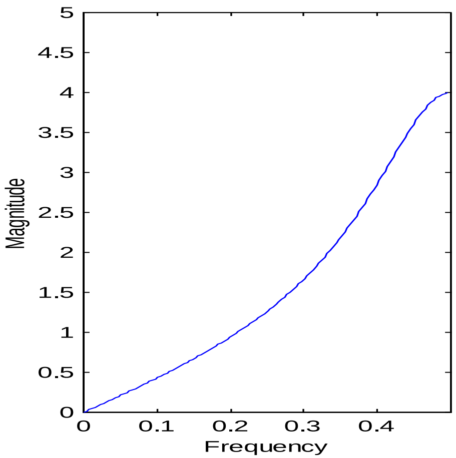
Application

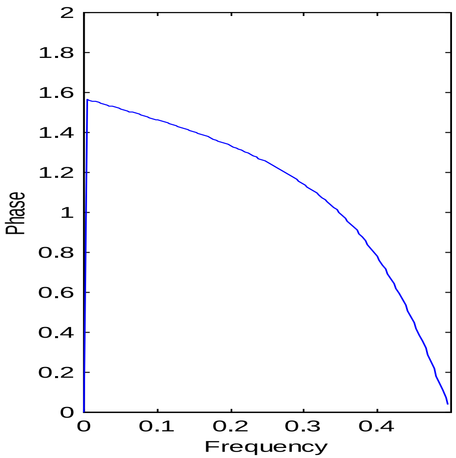


$$H(z) = \frac{1 - z^{-1}}{1 + 0.5z^{-1}}$$

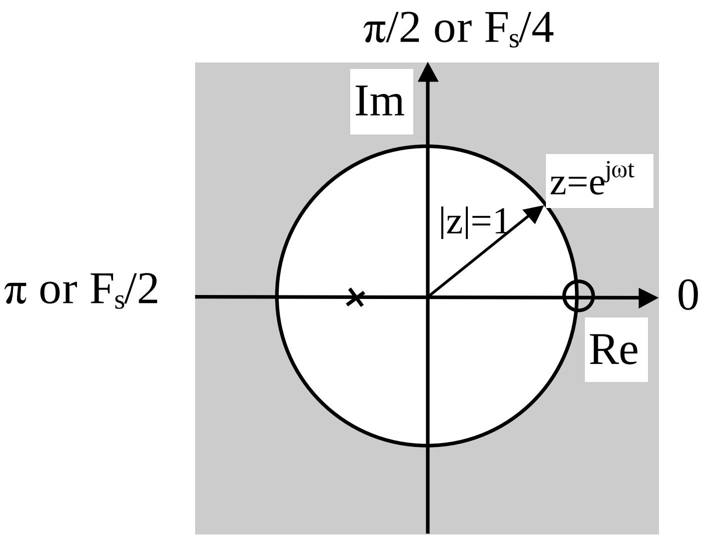
Stable

Frequency response
using MATLAB





Frequency on the Unit Circle



$$H(z) = \frac{1 - z^{-1}}{1 + 0.5z^{-1}}$$

The region from 0 to π corresponds to the region 0-F_s/2 of the Frequency response

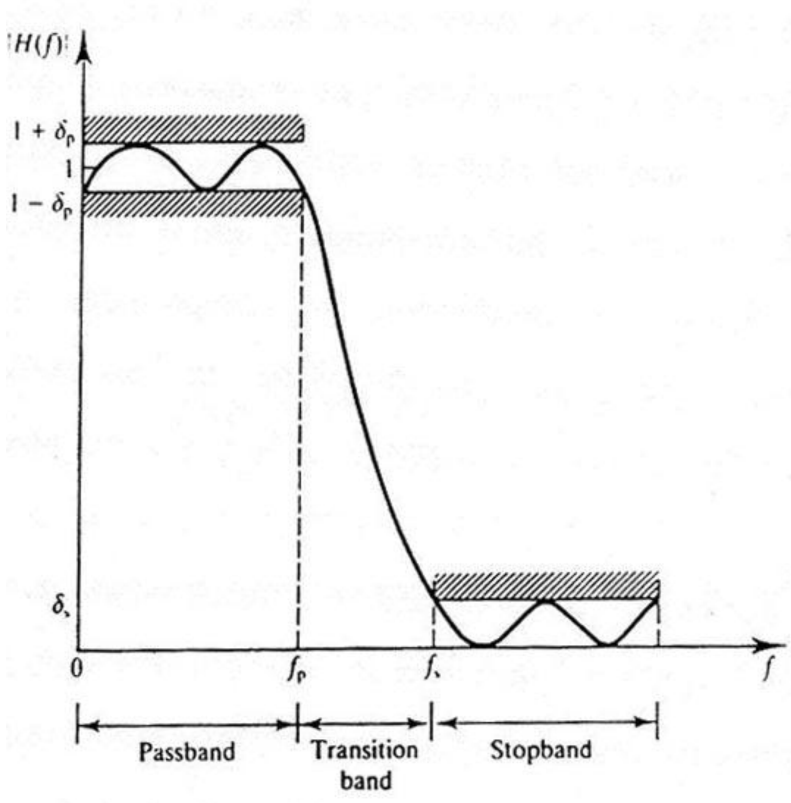
Digital Filters

•They are characterized by their Impulse Response h(n), their Transfer Function H(z) and their Frequency Response H(ω).

•They can have memory, high accuracy and no drift with time and temperature.

•They can possess linear phase.

•They can be implemented by digital computers.

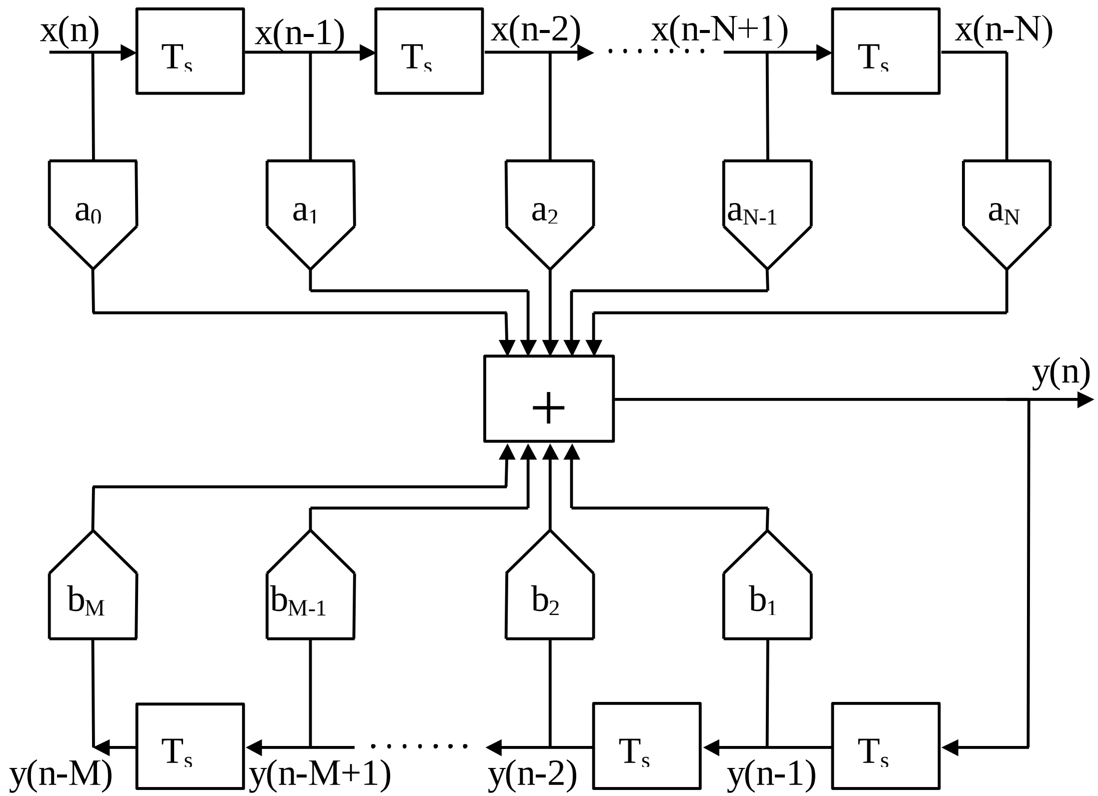


Digital Filters - Categories

IIR

$$y(n) = \sum_{k=0}^{N} a(k)x(n-k) - \sum_{k=1}^{M} b_k y(n-k)$$

$$H(z) = \frac{\sum_{k=0}^{N} a_k z^{-k}}{1 + \sum_{k=1}^{M} b_k z^{-k}}$$

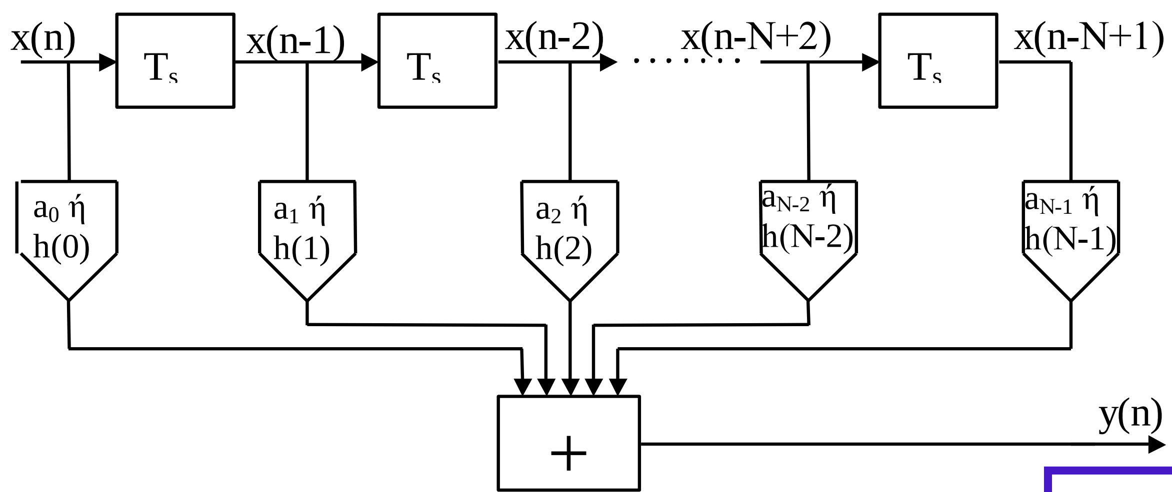


Digital Filters - Categories

FIR

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) = \sum_{k=0}^{N} a(k)x(n-k)$$

$$H(z) = \sum_{k=0}^{N} a_k z^{-k} = \sum_{k=0}^{N} h(k)z^{-k}$$



- Stable
- Linear phase

Digital Filters - Examples

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

a_0=0.498, a_1=0.927,
a_2=0.498, b_1=-0.674
b_2=-0.363.

$$H(z) = \sum_{k=0}^{11} h(k) z^{-k}$$

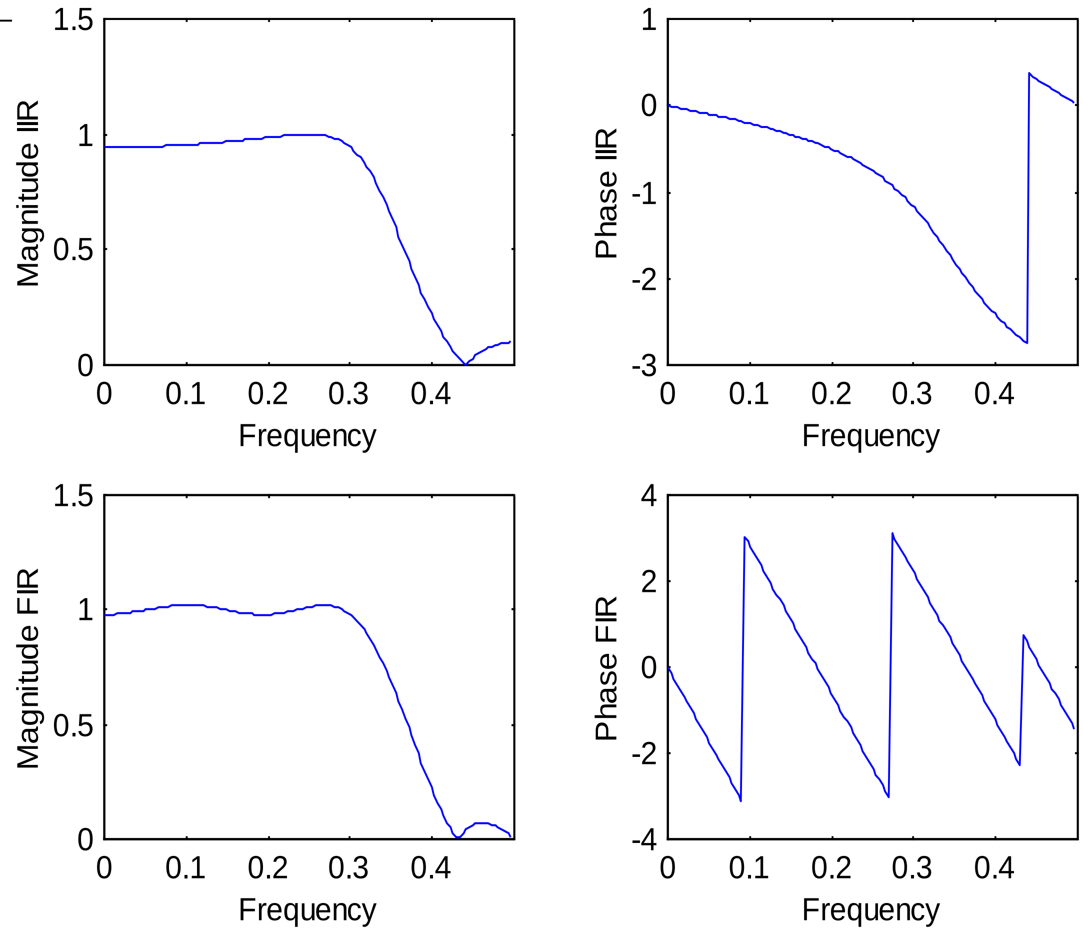
h(1)=h(10)=-0.04506

h(2)=h(9)=0.06916

h(3)=h(8)=-0.0553

h(4)=h(7)=-.06342

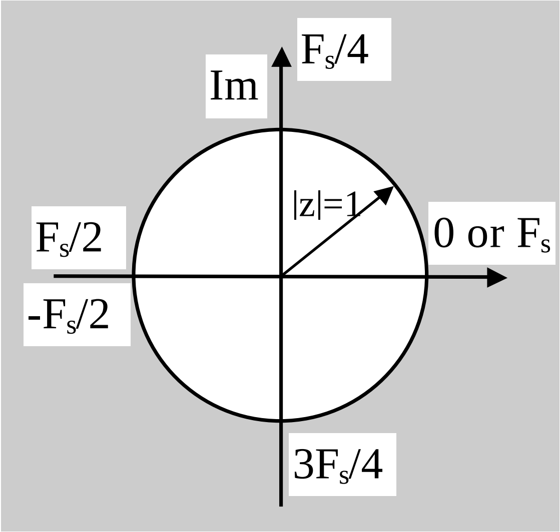
h(5)=h(4)=0.5789



IIR filter design

Design using the Bilinear *z*-transform, BZT

Design using the position of poles and zeros on the unit circle

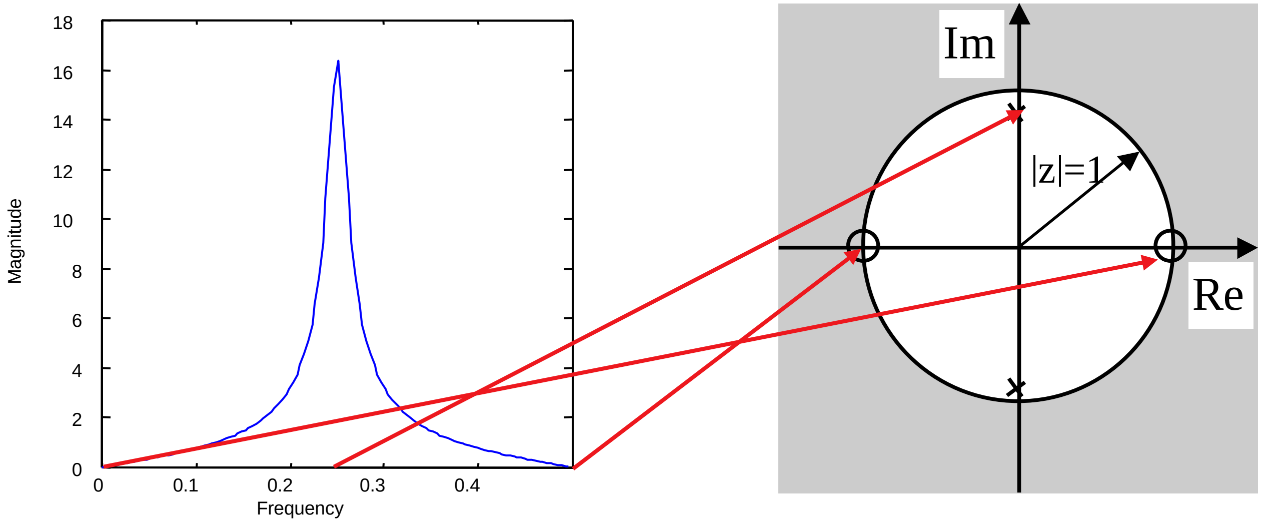


- The frequency response is zero at the points of zeros
- The frequency response takes a peak at the position of poles.
- In order to have real coefficients of the filter, the poles must appear in pairs. The same happens for the zeros as well.

IIR Filter Design - position poles and zeros on the unit circle

Design a band-pass filter with the following specifications

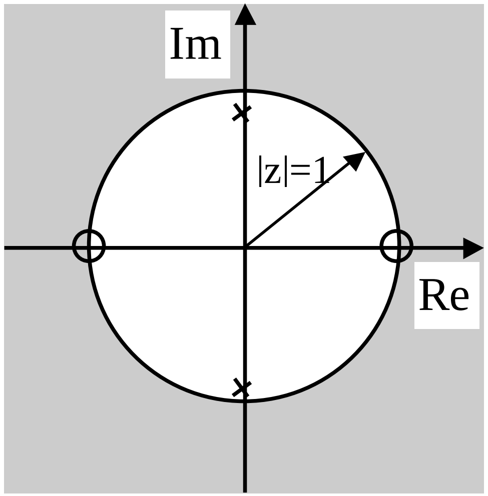
Sampling frequency 1000Hz, full rejection at dc and 500 Hz, Narrow pass-band at 250 Hz, 20Hz 3dBs bandwidth.

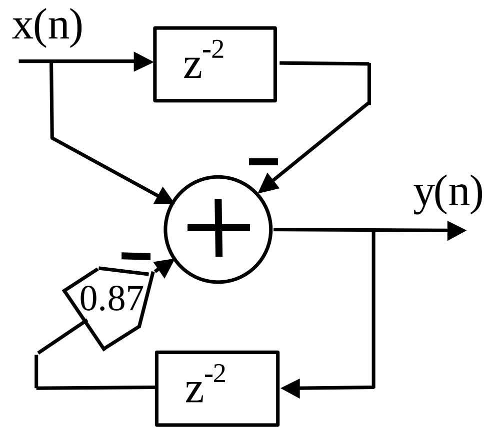


Zeros at 0 and $F_s/2$ or 180º (dc → 0, 500/1000→ $F_s/2$).
Poles at 250/1000→ $F_s/4$ or 90º and complex conjugate at -90º
The radius of poles should be smaller than 1 (stability)

$$r \approx 1-(bw/F_s)\pi \;=\; 1-(20/1000)\pi = 0.937$$

IIR Filter Design - position poles and zeros on the unit circle

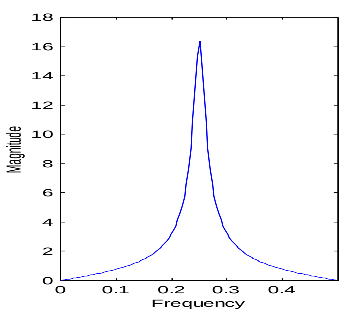


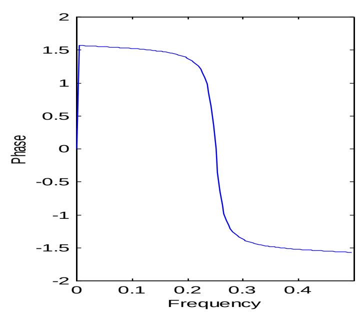


$$y(n) = x(n) - x(n-2) - 0.877969\, y(n-2)$$

$$r \approx 1-(bw/F_s)\pi \quad = 1-(20/1000)\pi = 0.937$$

$$H(z) = \frac{(z-1)(z+1)}{(z-re^{j\pi/2})(z-re^{-j\pi/2})} = \frac{z^2-1}{z^2+0.877969} = \frac{1-z^{-2}}{1+0.877969z^{-2}}$$

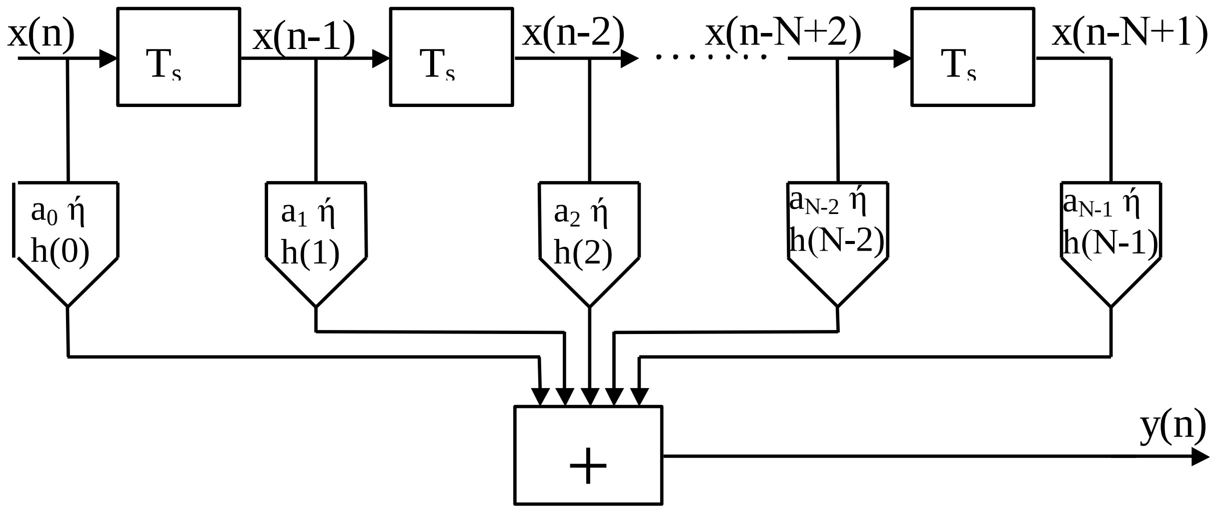




FIR filters

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) = \sum_{k=0}^{N} a(k)x(n-k)$$

$$H(z) = \sum_{k=0}^{N} a_k z^{-k} = \sum_{k=0}^{N} h(k) z^{-k}$$



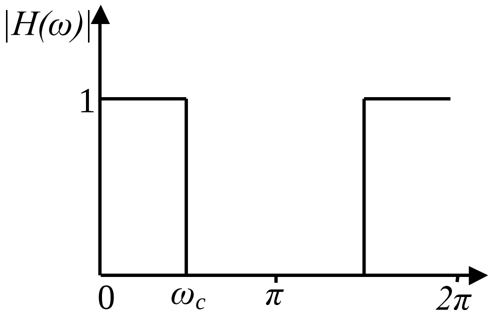
- Stable
- Linear phase

Design Methods

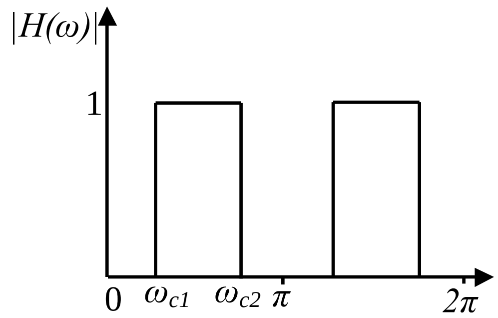
- Optimal filters
- Windows method
- Sampling frequency

Main categories of ideal filters

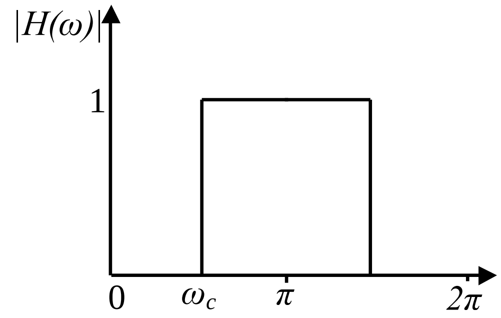
Low-pass

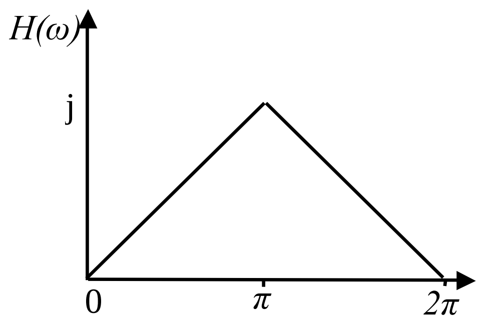


Band-pass

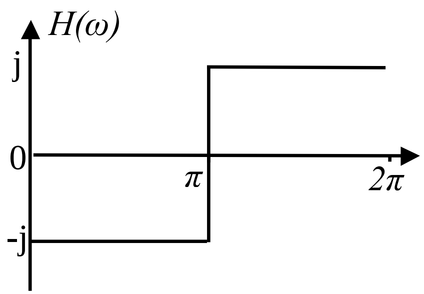


High-pass



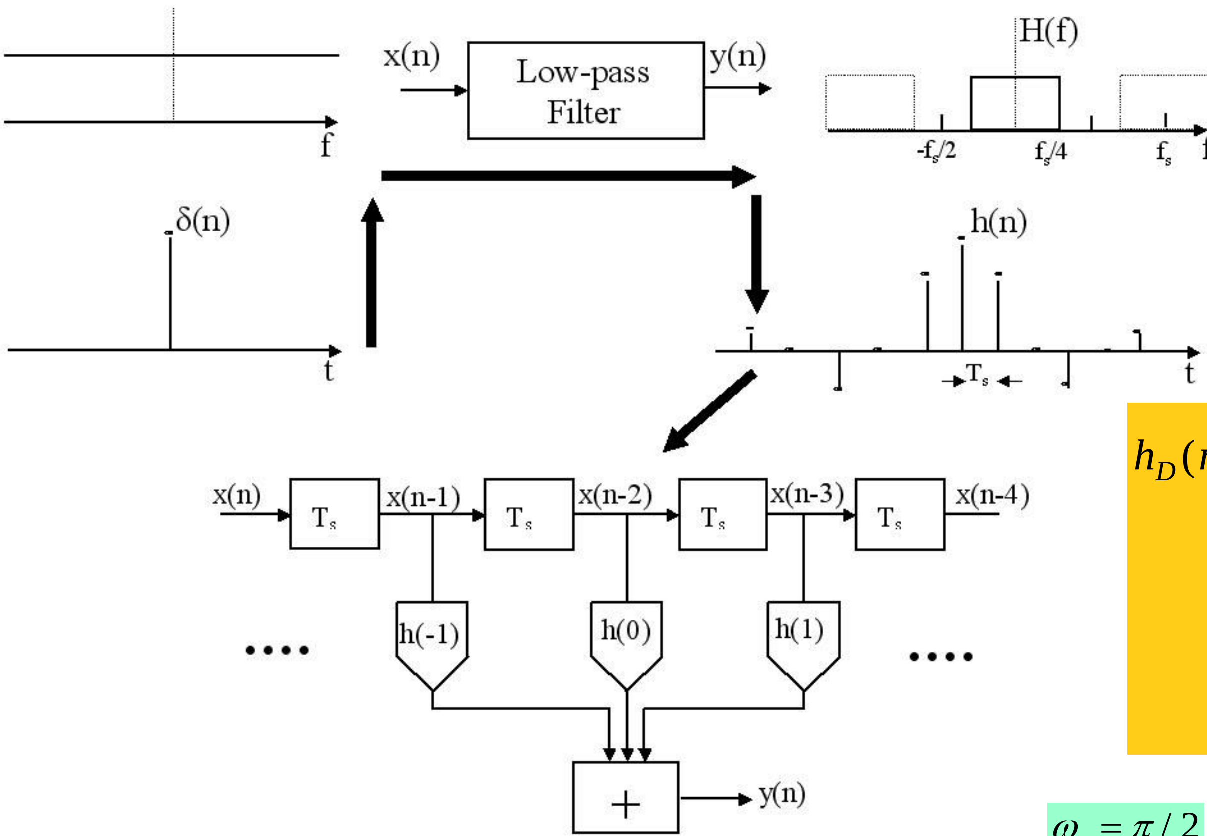


Differentiator



Hilbert Transformer

FIR filter design – basic concept



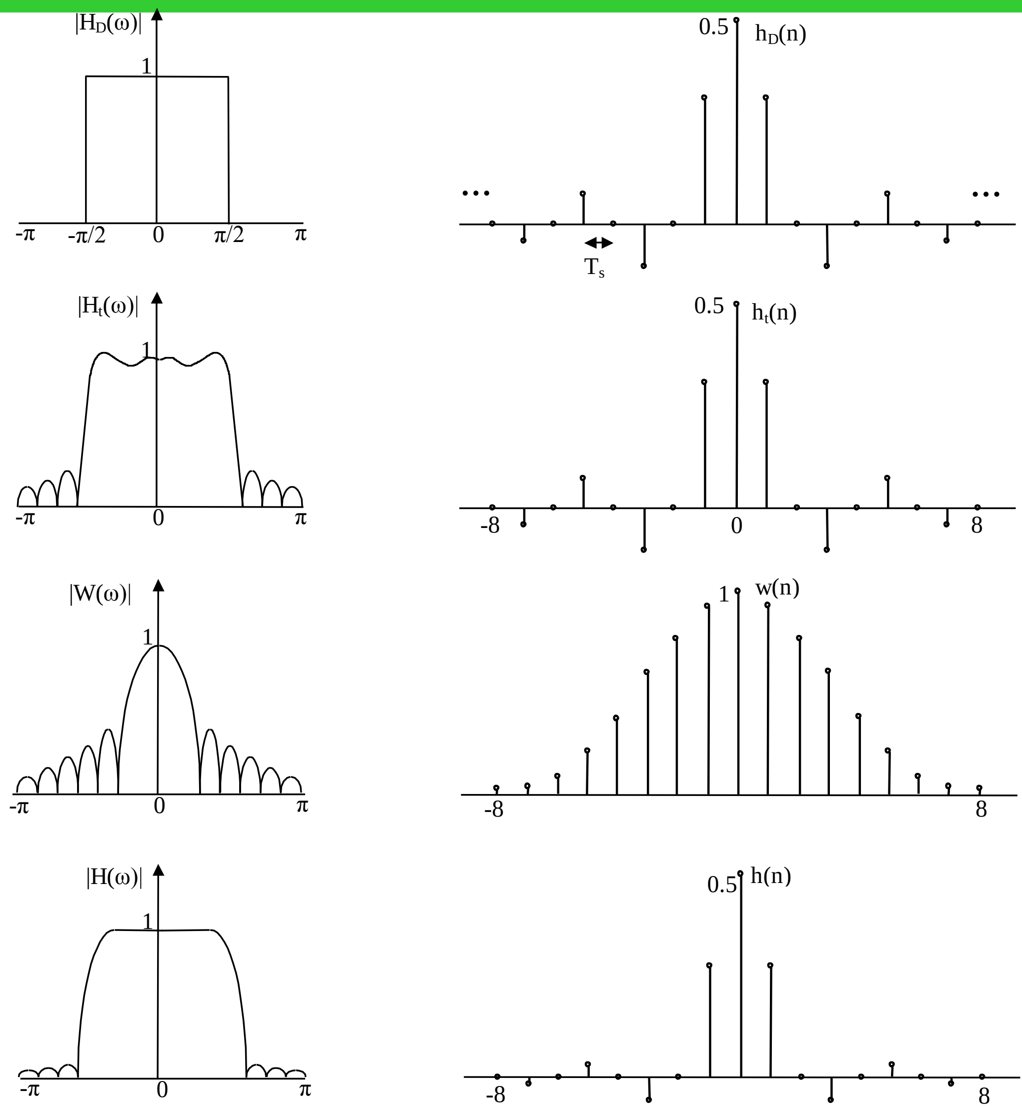
$$h_D(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_D(\omega)e^{j\omega n}d\omega =$$
$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega =$$
$$= \frac{2f_c \sin(n\omega_c)}{n\omega_c}$$

$$\omega_c = \pi/2$$

The filter coefficients result from the Inverse Fourier Transform of the Desired Frequency Response.

$$h(n) = \frac{1}{2}\frac{\sin(n\pi/2)}{(n\pi/2)}$$

FIR filter design – Windows method



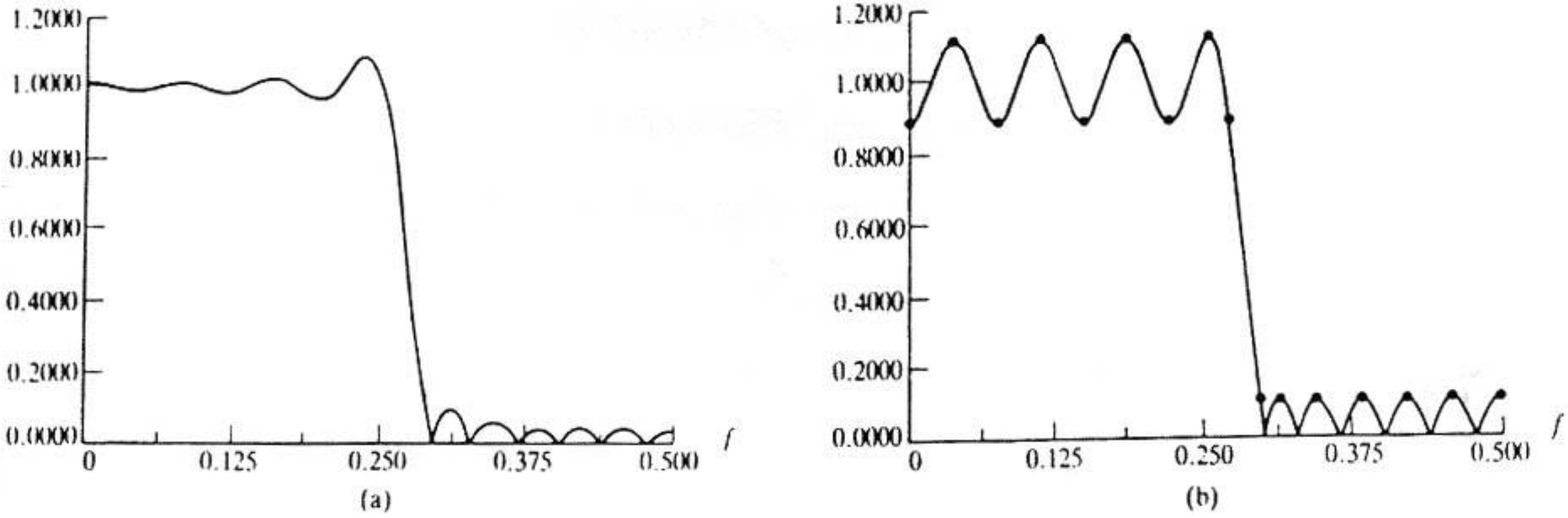
Gibbs phenomenon

Hamming Window

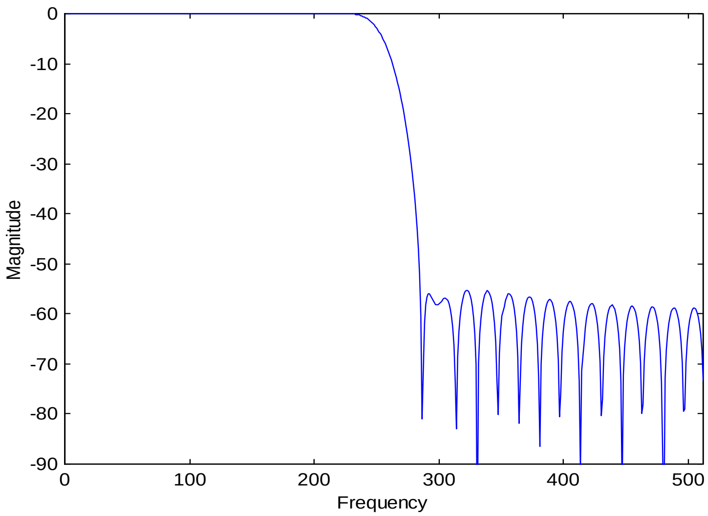
$$w_R(n) = \begin{cases} a + (1-a)\cos(\frac{2\pi n}{N}) \\ 0 \end{cases}$$

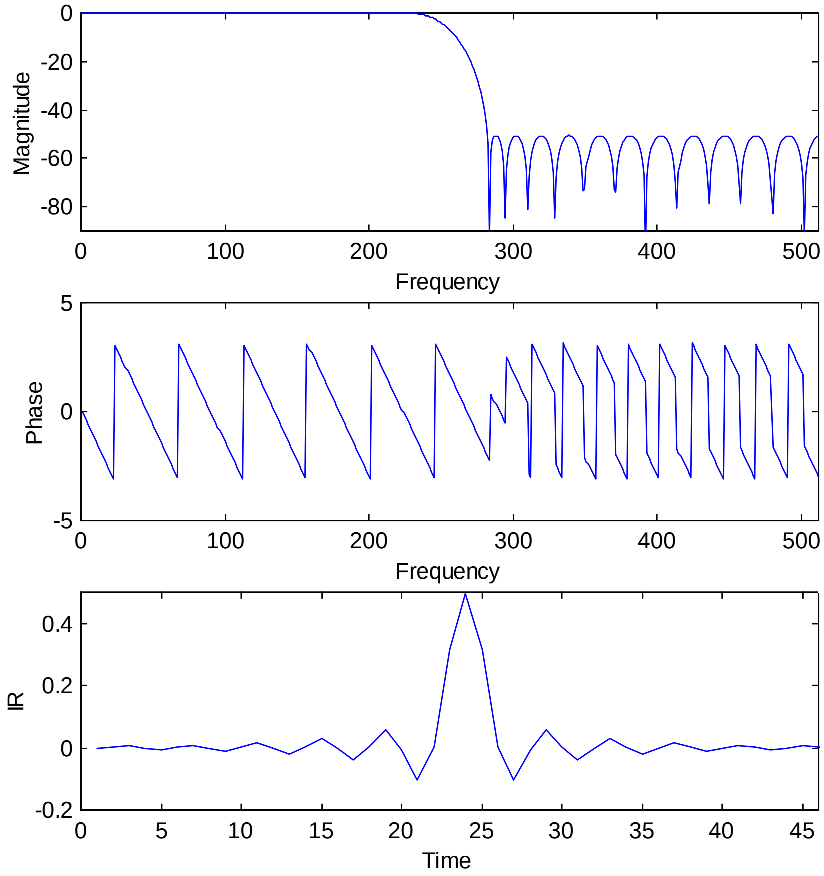
Parks and McClellan method.

Distributes Approximation error from the discontinuity all over the frequency band.



FIR filter design – Comparisons





Windows: 61 Coefficients

Parks and McClellan method: 46 and equiripple

FIR filters – Linear phase

Linear phase is strictly related with the symmetry of the Impulse Response

$$\theta(\omega) = -a\omega \qquad \theta(\omega) = b - a\omega$$

Condition $h(n) = h(N-n-1)$

$$H(\omega) = \sum_{k=0}^{6} h(k)e^{-j\omega kT} =$$

$$= h(0) + h(1)e^{-j\omega T} + h(2)e^{-j2\omega T} + h(3)e^{-j3\omega T} + h(4)e^{-j4\omega T} + h(5)e^{-j5\omega T} + h(6)e^{-j6\omega T}$$

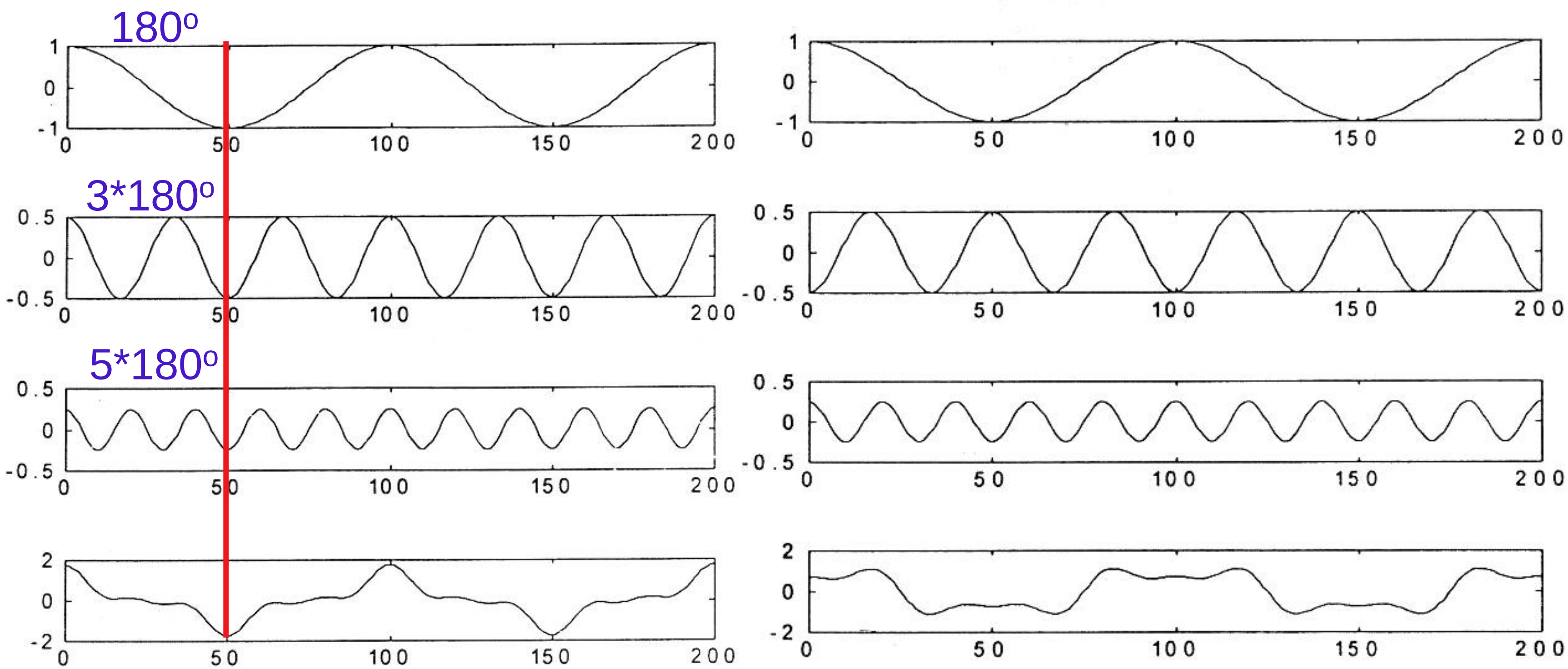
$$= e^{-j3\omega T}\left[h(0)e^{j3\omega T} + h(1)e^{j2\omega T} + h(2)e^{j\omega T} + h(3) + h(4)e^{-j\omega T} + h(5)e^{-j2\omega T} + h(6)e^{-j3\omega T}\right]$$

$$= e^{-j3\omega T}\left[h(0)(e^{j3\omega T} + e^{-j3\omega T}) + h(1)(e^{j2\omega T} + e^{-j2\omega T}) + h(2)(e^{j\omega T} + e^{-j\omega T}) + h(3)\right]$$

$$= e^{-j3\omega T}\left[2h(0)\cos(3\omega T) + 2h(1)\cos(2\omega T) + 2h(2)\cos(\omega T) + h(3)\right]$$

The phase is introduced by the term $e^{-j3\omega T}$ and equals
$\theta(\omega)=3\omega T=\omega T\,(7-1)/2$

Linear phase - Same time delay



If you shift in time one signal, then you have to shift the other signals the same amount of time in order the final wave remains unchanged.

For faster signals the same time interval means larger phase difference.

Proportional to the frequency of the signals. $-\alpha\omega$